Qualifying Exam Proposal
Forced synchronization - a study in control of complex systems

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Date:

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Abstract

Complex systems are ubiquitous, and often difficult to control. As a toy model for the control of a complex system, we take a system of coupled phase oscillators, all subject to the same periodic driving signal. It has been shown previously that in the absence of coupling, this can result in each oscillator attaining the frequency of the driving signal, with a phase offset determined by the oscillator's natural frequency. We consider a special case in which the coupling tends to destabilize the phase configuration to which the driving signal would send the oscillators in the absence of coupling. In this setting we derive stability estimates that capture the trade-off between driving and coupling, and compare these results to the unforced version (i.e. the standard Kuramoto model).

Part I
Background

1 Motivation

A system is called *complex* if it exhibits surprising properties due to the structured interactions of its components. Another term for this property is *self-organizing.* Examples of complex systems and some of the emergent phenomena they can exhibit include:

1. The brain: epileptic seizures [8]
2. The world economy: financial collapse [14]
3. The global climate: rapid climate change [11]
4. Ecosystems: extinction events/algae blooms [10]
5. Social networks: memes [1]
6. The power grid: rolling blackouts [6]
7. Schooling/swarming/herding animals: school formation/dissolution [2]
8. Spatially extended chemical reactions (e.g. Belousov-Zhabotinsky): formation of spatial patterns out of homogeneous background [4]

In many cases we would like the ability to predict, or even control, the onset of a certain collective phenomenon. To that end, it is both theoretically and practically interesting to understand the fundamental limits of control of a self-organizing system.

A theme running through many examples of collective phenomena is order winning out over disorder. For example, in the case of a flock of birds, each bird follows its own path - however, they have enough impetus to follow each other that the entire flock moves as one. In the case of financial collapse, each publicly traded company has its own expenses and revenue streams, but are dependent enough on each other that a large enough failure of a few companies can trigger a large-scale collapse.

There are situations in which global order is desirable for proper functioning of a system, such as the very precise phase synchronization of electric currents in the power grid; there are also situations in which global order has disastrous consequences, such as massively synchronized firing of neurons in an epileptic seizure, or mechanical resonance (such as aerodynamic flutter). In still other situations, the value may be ambiguous, such as the schooling and de-schooling of a collection of fish, or the meteoric rise of a socio-political movement.

The key features I wish to focus on with respect to control of complex systems are the trade-offs between order and disorder. To this end, I propose to study a system of forced, coupled, nonlinear oscillators, for reasons which I outline below.

2 Basic formalism and standard results

2.1 Entrainment

The first reason to consider a system composed of nonlinear oscillators is that their response to simple control signals is both intricate and analytically tractable. In particular, they exhibit *entrainment*, which is when an oscillator moves with the frequency of the control signal, rather than its natural frequency.

References [18, 19] and related work study a model of a forced phase oscillator governed by the dynamics

$$\dot{\psi} = \omega + Z(\psi)u$$

where ω denotes the natural frequency, u an external driving signal, and Z the *phase response curve*, which defines the response of the oscillator to the external driving. This model can be derived as an approximation to a nonlinear limit cycle oscillator in the regime of small forcing amplitude (see, e.g. [12]). If we choose $u = v(\Omega t)$, where $\Omega \sim \omega$ and v is 2π-periodic, then we can approximate the dynamics of ψ by shifting to a reference frame moving with frequency Ω and averaging the influence of the driving signal over a whole driving period, and obtain

$$\dot{\varphi} = \Delta\omega + \Lambda_v(\varphi) \tag{1}$$

where $\Delta\omega := \omega - \Omega$ is the *frequency detuning*, and $\Lambda_v(\varphi) = \int_0^{2\pi} Z(\varphi + \theta)v(\theta)d\theta$ is the *interaction function*. It can be shown that φ is a good approximation to $\psi - \Omega t$ as $t \to \infty$ [12].

The upshot of (1) is that provided $\Delta\omega + \Lambda_v(\varphi) = 0$ has a solution φ^* and that $\Lambda_v'(\varphi^*) < 0$, φ will settle down to φ^* as $t \to \infty$ (given appropriate initial conditions; see figure 1). This means that in certain cases, we can design an *open-loop control policy* (that is, one without feedback) to drive an oscillator to a known phase offset of the control signal.

Moreover, this φ^* depends on ω. So, if we have a population of non-identical oscillators (that is, different ω but the same Z), we can obtain a diversity of phase offsets by using a single driving signal $u = v(\Omega t)$. In the context of order vs. disorder, then, the nature of this sort of control is somewhat dual. On one hand, all the oscillators obtain the same frequency; on the other hand, they obtain different phases. One contribution we make is to quantitatively analyze the effect of one such control signal on the onset of collective synchronization.

2.2 Synchronization

Synchronization is a classic example of collective behavior, and one which connects nicely to the entrainment behavior described above. The so-called *Kuramoto model* [7] is arguably the simplest known system exhibiting a synchronization transition. In its original form, it consists of a large population of non-identical phase oscillators coupled uniformly to each other by the sine function. That is, the model reads

$$\dot{\varphi}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\varphi_j - \varphi_i)$$

where $\{\varphi_i\}$ are the phases of the N oscillators, $\{\omega_i\}$ their respective natural frequencies, and $K \in \mathbb{R}$ represents the overall coupling strength.

A classic result about the Kuramoto model comes from passing to the limit of infinitely many oscillators, and treating the state of the system not as a configuration of phases, but as a probability

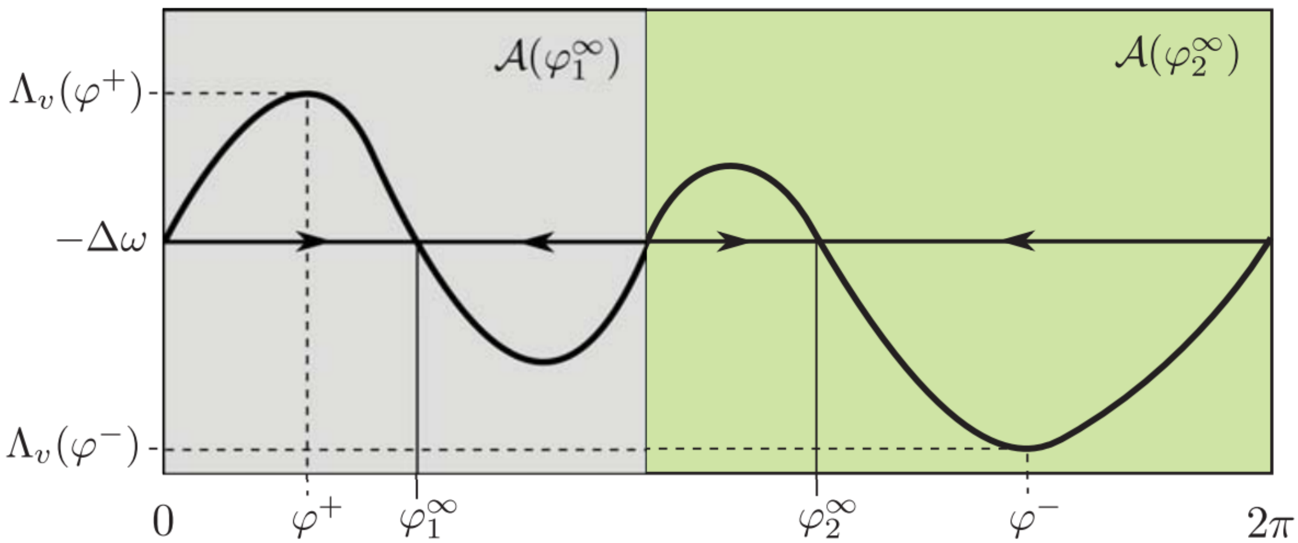

Figure 1: Example interaction function for a forced nonlinear oscillator. The sets $\mathcal{A}(\varphi_1^\infty)$ and $\mathcal{A}(\varphi_2^\infty)$ are basins of attraction for the asymptotic phase values φ_1^∞ and φ_2^∞, given frequency detuning $\Delta\omega$. Taken from SI of [19]

distribution over values of (ω, φ). Given the natural frequencies distributed according to a unimodal density $g(\omega)$ centered (WLOG) at $\omega = 0$, it can be shown [15] that there is a "phase transition" at a coupling strength

$$K = K_c = \frac{2}{\pi g(0)}$$

That is, when $K < K_c$, the system settles down to an incoherent state, where the phase distribution is spread evenly around the unit circle. At $K > K_c$, the incoherent state is no longer stable, and the system settles into a state where a subset of oscillators are synchronized with each other, and the phase distribution obtains a singular (i.e. delta-function-like) piece.

3 Challenges and directions forward

The phase transition found in the Kuramoto model arises from a trade-off of intrinsic disorder (i.e. the distribution of natural frequencies $g(\omega)$) and interactions which drive the system to order (i.e. the coupling terms $K\sin(\varphi_i - \varphi_j)$). On the other hand, the entrainment phenomena discussed in [18, 19] are the effect of a force which may drive the system to a state which is ordered, but whose order is different than the order arising from coupling. Explicitly, applying a driving signal to a heterogeneous population of oscillators will, in the absence of coupling, lead in general to unequal phases. I propose to study the interplay of entrainment via driving and synchronization via coupling.

3.1 Model description and methods

The following is based on research conducted during the summer of 2016 at the Center for Nonlinear Studies, Los Alamos National Laboratory, in collaboration with Aric Hagberg and Anatoly Zlotnik.

I propose to study models of the form

$$\dot{\varphi}_i = \omega_i + \Lambda_v(\varphi_i) + K \sum_{j=1}^{N} A_{ij} G(\varphi_j - \varphi_i) \tag{2}$$

where $A = (A_{ij})$ is a matrix describing the coupling of individual oscillators, G the coupling function, K represents the overall coupling strength, and Λ_v is the interaction function as described above, which depends on the phase response curve Z and the driving signal v.

Given particular choices of Λ_v and g, I will investigate existence and linear stability of fixed points in terms of the parameters K and (A_{ij}), both for finite N and in the limit $N \to \infty$.

The mathematical description of the $N \to \infty$ limit follows the tradition of much of the seminal work on the Kuramoto model, e.g. [17, 9, 7]. That is, we consider as a "state" of the system a family of probability distributions (ρ_ω) over the unit circle S^1, indexed by natural frequency ω, which give the density of oscillators with a given frequency ω and phase φ. Conservation of the number of oscillators with each natural frequency leads to a continuity equation:

$$\partial_t \rho_\omega + D(v_\omega \rho_\omega) = 0 \tag{3}$$

where D denotes the derivative in the sense of distributions, and $v_\omega = v_\omega(\varphi)$ is the phase velocity - that is, $\dot{\varphi}$ for an oscillator with natural frequency ω and phase φ. If we take the coupling to be all-to-all, the phase velocity reads

$$v_\omega = \omega + \Lambda_v(\varphi) + K \int_\Omega d\mu(\omega') \int_{S^1} d\rho_{\omega'}(\varphi') G(\varphi' - \varphi)$$

where $\mu \in \mathcal{P}r(\Omega)$ denotes the distribution of natural frequencies over the range of possible frequencies Ω. Hence, equation (3) constitutes a nonlinear partial integro-differential equation for the state $\rho = (\rho_\omega)$. Analogously to the finite-dimensional case, we can find fixed points by setting $\partial_t \rho_\omega = 0$ for (almost) all $\omega \in \Omega$, linearize the dynamics around such fixed points, and find the spectra of the resulting linear operators to draw conclusions about the stability of these fixed states. This technique was pioneered by Strogatz and Mirollo in [17], where it was used specifically to analyze the stability of the incoherent state (i.e. $d\rho_\omega = \frac{1}{2\pi} d\varphi$ for all ω).

One qualitative difference to note between the finite- and infinite-dimensional frameworks is that in the former, a fixed point satisfies $\dot{\varphi}_i = 0$ for all i. However, a fixed point of the infinite-dimensional dynamics may have $v_\omega \neq 0$, corresponding to ρ_ω that are absolutely continuous with respect to Lebesgue measure on S^1. In other words, a "fixed state" of the infinite-dimensional dynamics can describe *drifting* oscillators, since the state is only "fixed" in a statistical sense. We should therefore be careful in our comparison of the two versions of the dynamics.

Part II

Proposed Research

4 Mean-field coupling meets perfect decoherence

To extremize the competition between entrainment and coupling, I will first consider the case where $G = \sin$, $A_{ij} = \frac{1}{N}$ (that is, mean-field attractive coupling), and Λ_v is designed so that the entrained phases $\{\varphi_i^*\}$ of the uncoupled system are spread evenly around the unit circle. This can be most easily achieved if $\omega_i = \frac{2i}{N} - 1,\ i = 1, \ldots, N$, and

$$\Lambda_v(\varphi) = \frac{-\varphi}{\pi}, \quad \varphi \in (-\pi, \pi]$$

so that $\varphi_i^* = \pi\omega_i = \frac{2\pi i}{N} - \pi$ (see figure 2).

In the continuum limit, $\omega_i = \frac{2i}{N} - 1$ is replaced by the density $g(\omega) = \frac{1}{2}\chi_{(-1,1]}$

4.1 Partial results

In finite dimensions, one can derive bounds on the eigenvalues of the Jacobian via the Gershgorin circle theorem. The most general bound states that if the coupling strength K satisfies the inequality

$$K < \frac{-\Lambda_v'(\varphi_i^*)}{2k_i\|G'\|_\infty}$$

then the fixed point $\varphi^* = (\varphi_i^*)$ is linearly stable. Here $\|\cdot\|_\infty$ denotes the supremum norm, and $k_i = \sum_j A_{ij}$ is the total (weighted) degree of oscillator i.

In the special case of global sine coupling (i.e. $G = \sin$ and $A_{ij} = \frac{1}{N}$) and interaction function $\Lambda_v'(\varphi) = \frac{-\varphi}{\pi}$, which drives the system towards desynchronization, we have that the desynchronized phase configuration $\varphi_i^* = \frac{2\pi i}{N} - \pi$ is linearly stable provided $K < \frac{1}{\pi}$ (which is an improvement on the generic bound above by a factor of 2).

In infinite dimensions, we can consider the case of global sine coupling with interaction function $\Lambda_v'(\varphi) = \frac{-\varphi}{\pi}$, and again linearize about the desynchronized fixed point. In this case, using machinery developed extensively in [9], we can in fact exactly diagonalize the linearized dynamics. We find that the eigenvalues are $-\frac{1}{\pi}$ and $-\frac{1}{\pi} + \frac{K}{2}$, which implies that the desynchronized fixed point is linearly stable if $K < \frac{2}{\pi}$ and linearly unstable if $K > \frac{2}{\pi}$. By virtue of the exact diagonalization, we can go further and describe the *modes* (i.e. eigenvectors) that go unstable (but we omit this for now).

One important consequence is that we have found a critical point $K_c = \frac{2}{\pi}$, which is in fact *smaller* than that for the unforced Kuramoto model; with the same phase distribution, uniform on $(-1, 1]$, the unforced Kuramoto model has a critical point $K_c = \frac{2}{\pi g(0)} = \frac{4}{\pi}$. In other words, it is easier to bring the system to phase synchronization when it is driven in such a way as to spread the phases evenly across the unit circle. The difference, of course, is that in the driven system, the oscillators have already been brought to frequency synchronization - in some sense, “halfway” to phase synchronization.

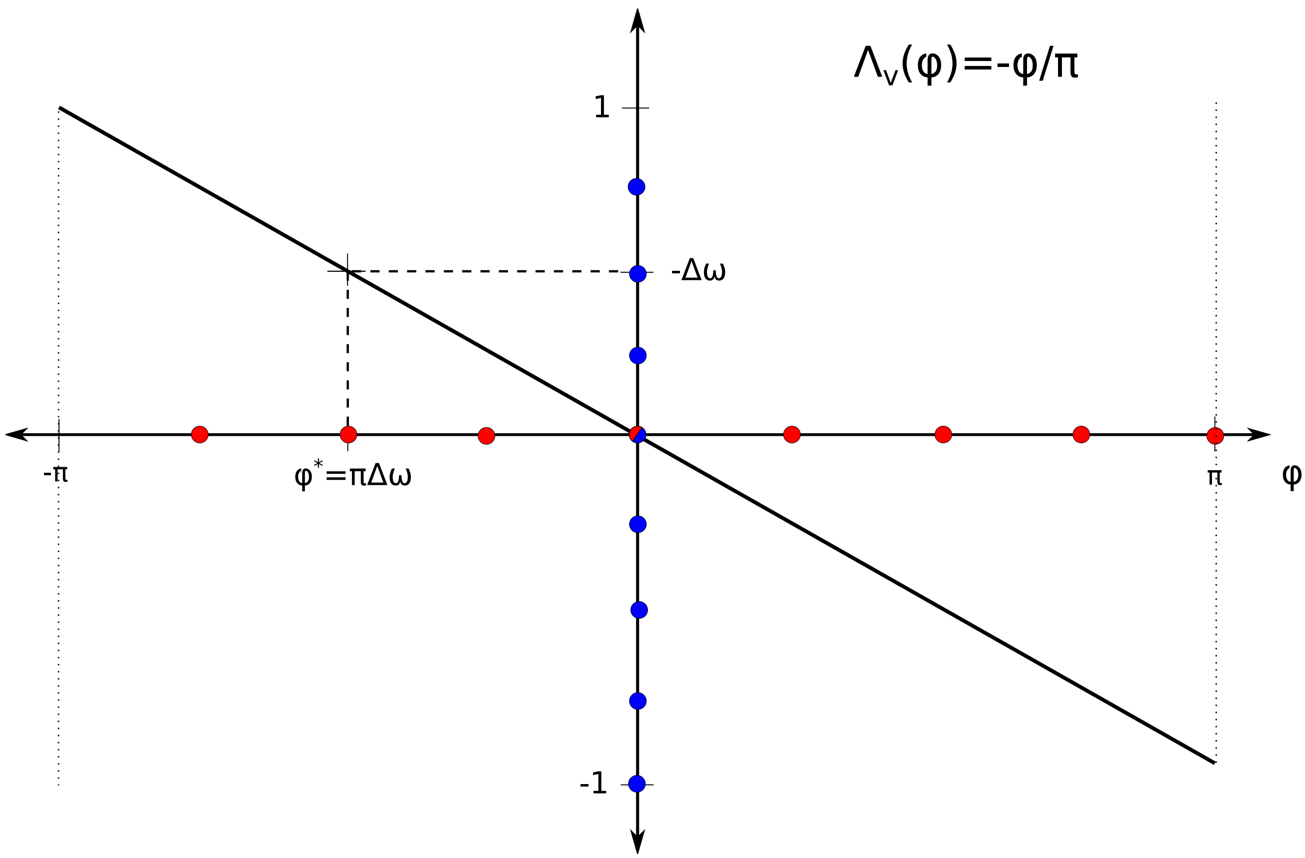

Figure 2: Interaction function for perfect desynchronization

4.2 Future work

During the Fall 2016 quarter, I plan to perform simulations to corroborate my results for finite N and to get a sense for convergence as $N \to \infty$ to the expected mean-field behavior. I plan to collect these results into a paper to be submitted to *Nonlinearity* during Winter quarter 2017.

5 Structured Coupling

It's well-known that the way in which oscillators are coupled can have profound effects on the manner in which the population synchronizes. For example, it is found in [16] that the probability of a macroscopic synchronized cluster of oscillators decays exponentially with system size if the coupling topology is any finite-dimensional lattice.

Other work, such as [5], has focused on the case of a fixed, finite number of oscillators with arbitrary coupling topology, obtaining existence, uniqueness, and stability results for the finite-dimensional ODE directly. These results give bounds on a critical coupling strength in terms of topological properties of the coupling, such as the degree distribution and eigenvalues of the graph Laplacian.

My contribution will be to explain how the relationship between critical coupling strength and network topology changes with the introduction of a common periodic driving signal. Based on the intuition I've built so far, I expect that global entrainment will generally bring networks of coupled oscillators closer to synchrony, and to eliminate the possibility of chaos, as is seen, for instance, in [13].

To achieve this goal, I will first gain intuition by numerical solution of the ODE (2) for various adjacency matrices (A_{ij}). I will then seek to explain my numerical results by extending the dynamical systems-based approach of [5], and complement this with an appropriate statistical treatment along the lines of the work I have already done.

I plan to address these problems during the Winter and Spring quarters of 2017, and collect the results into a publication to be submitted in early summer 2017.

6 Multiple Timescales

An extensive theory was developed in [18, 3] for *subharmonic* entrainment of phase oscillators. That is, an oscillator is driven by a signal at frequency $N\Omega$, and the oscillator settles into a motion with frequency $M\Omega$, where $N, M \in \mathbb{Z}$ and $N \neq M$. This sort of resonance phenomenon is well-know in the physics literature. However, it is not well understood what can happen when many subharmonically forced oscillators are coupled to one another.

This sort of situation arises naturally in many real systems. For instance, the human body is a composite of very many intricately interconnected units, many of which undergo repetitive motion on a wide range of timescales, from neural firing in the brain, to the beating of the heart, to breathing, to eating, to the circadian rhythm, and beyond. Moreover, the whole system is subjected to external stimulus of different periods as well, such as the day/night cycle, to the work week, to the annual cycle of the seasons.

My goal in this task, therefore, is to understand the ways in which repetitive motions at different timescales can interfere with and reinforce each other, both in isolation and when subject to common forcing.

As a first model to consider, we could imagine a population of oscillators split into two groups: fast and slow. The fast oscillators have natural frequencies distributed around some characteristic frequency Ω_1, while the slow oscillators have natural frequencies distributed around $\Omega_2 = \frac{1}{2}\Omega_1$. We could then ask the question: is it possible to observe all fast oscillators attain frequency Ω_1 and all slow oscillators attain frequency Ω_2 by only introducing coupling? Next, is this sort of frequency locking made more probable by application of a common driving signal?

I intend to treat problems of this sort starting in the summer of 2017, possibly returning to Los Alamos to collaborate with Anatoly Zlotnik. This work will continue into the 2017-2018 academic year, culminating in a publication to be submitted around Winter/Spring 2018.

7 Proposed Timeline

Fall 2016 Complete Qualifying Exam; perform simulations; write and submit first paper.

Winter 2017 Begin analysis of structured coupling

Spring 2017 Continue analysis of structured coupling and draft second paper

Summer 2017 Submit second paper; return to Los Alamos and begin analysis of multiple timescales

Fall 2017 Continue work on multiple timescales; begin writing dissertation

Winter 2018 Draft paper on multiple timescales; continue writing dissertation

Spring 2018 Submit third paper; complete dissertation

Part III
Syllabus

1. Analysis (MAT 201ABC)
   (a) Banach and Hilbert spaces
   (b) Measure theory

2. Applied Math (MAT 207ABC)
   (a) Dynamical Systems/Bifurcation theory
   (b) Solution of ODEs & PDEs

(c) Perturbation theory

3. Probability and Stochastic Processes (MAT 235AB, 236A)
   (a) Central Limit Theorem
   (b) Brownian Motion
   (c) Stochastic integration
4. Information Theory (PHY 256AB)
5. Network Theory (MAE/ECS 253)
   (a) Dynamics on Networks
   (b) Network formation processes
   (c) Random graphs

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