Qualifying Exam Syllabus Proposal

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Date:
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Time:
11AM-2PM
Location:
MSB 3106

1 Proposed Research Talk

1.1 The Catalan numbers and q-analogues.

A lattice path is a sequence of North $N(0,1)$ and East $E(1,0)$ steps in the first quadrant of the xy-plane, starting from the origin $(0,0)$ and ending at say (n,m). We let $L_{n,m}$ denote the set of all such paths, and $L^+_{n,m}$ the subset of $L_{n,m}$ consisting of paths which never go below the line $y = \frac{m}{n}x$. A rational Dyck path is an element of $L^+_{n,m}$ for some n,m.
Let $C_{n,m} = \frac{1}{n+m}\binom{n+m}{n}$ denote the rational Catalan number. For coprime n and m, $C_{n,m}$ also counts the number of elements in $L^+_{n,m}$. For the majority of the talk we will only be interested in the special case $m = n+1$, so that $C_{n,n+1} = C_n = \frac{1}{n+1}\binom{2n}{n}$ is the usual nth Catalan number.
There is a useful recursive relation between Catalan numbers:

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}, \quad n \geq 1. \tag{1}$$

Given $\pi \in L^+_{n,m}$, let σ be the 0,1-string resulting from the following algorithm. First initialize σ to the empty string. Next start at $(0,0)$, move along π and add a 0 to the end of $\sigma(\pi)$ every time a N step is encountered, and add a 1 to the end of $\sigma(\pi)$ every time an E step is encountered. We call the transformation of π to σ or its inverse the *coding* of π or σ. Denote the major index statistic of the string σ to be

$$\mathrm{maj}(\sigma) = \sum_{i:\sigma_i > \sigma_{i+1}} i.$$

Now let $a_i(\pi)$ denote the number of complete squares, in the ith row from the bottom of π, which are to the right of π and to the left of the line $y = \frac{m}{n}x$. We set $\mathrm{area}(\pi) = \sum_i a_i(\pi)$.

In sections 1.2-1.7, we will be looking at q- and q,t- generalizations of the usual Catalan numbers C_n. First, we define q- analogues for binomial coefficients. Let

$$[n] = \frac{q^n - 1}{q-1}, \quad [n]! = [1][2]\dots[n], \quad \begin{bmatrix} n+m \\ m \end{bmatrix} = \frac{[n+m]!}{[n]![m]!}.$$

The first natural q-analogue of C_n is given by the following theorem:

Theorem 1.1 *(MacMahon[Mac60])*

$$\sum_{\pi \in L^+_{n,n}} q^{maj(\sigma(\pi))} = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

The second natural q-analogue was studied by Carlitz and Riordan [CR64]. They define

$$C_n(q) = \sum_{\pi \in L^+_{n,n}} q^{\text{area}(\pi)}.$$

Proposition 1.2

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_k(q) C_{n-k}(q), \quad n \geq 1.$$

1.2 Hilbert and Frobenius series.

Given any subspace $W \subseteq \mathbb{C}[X_n, Y_n] = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, we define the bigraded Hilbert series of W as

$$\mathcal{H}(W; q, t) = \sum_{i,j \geq 0} t^i q^j \dim(W^{(i,j)}),$$

where the subspaces $W^{(i,j)}$ consist of those elements of W of bi-homogeneous degree i in the x variables and j in the y variables. Also, define the diagonal action of S_n on W by

$$\sigma f = f(x_{\sigma_1}, \ldots x_{\sigma_n}, y_{\sigma_1}, \ldots y_{\sigma_n}), \quad \sigma \in S_n, \quad f \in W.$$

Irreducible characters of S_n are in one-to-one correspondence with partitions $\lambda \in \text{Par}(n)$. We denote them as χ^λ.

The diagonal action fixes the subspaces $W^{(i,j)}$, so we can define the bigraded Frobenius series of W as

$$\mathcal{F}(W; q, t) = \sum_{i,j \geq 0} t^i q^j \sum_{\lambda \vdash n} s_\lambda \text{Mult}(\chi^\lambda, W^{(i,j)}).$$

Similarly, let W^ε be the subspace of alternating elements in W, and

$$\mathcal{H}(W^\varepsilon; q, t) = \sum_{i,j \geq 0} t^i q^j \dim(W^{\varepsilon(i,j)}).$$

It's a known fact that

$$\mathcal{H}(W^\varepsilon; q, t) = \langle \mathcal{F}(W; q, t), s_{1^n} \rangle.$$

1.3 Partitions.

A partition λ is a nonincreasing finite sequence $\lambda_1 \geq \lambda_2 \geq \ldots$ of positive integers. We call each λ_i a part. Let $l(\lambda)$ denote the number of parts and $|\lambda| = \sum_i \lambda_i$ the sum of the parts. If λ is a partition and $|\lambda| = n$, we also say $\lambda \vdash n$ or $\lambda \in \text{Par}(n)$. The *Ferrers graph* of λ is an array of unit squares, called cells, with λ_i cells in the ith row, with the first cell in each row left-justified. We define the conjugate partition, λ' as the partition of those Ferrers graph is obtained from λ by reflecting across the diagonal $x = y$. For example, $(i, j) \in \lambda$ refers to a cell with $(column, row)$ coordinates (i, j), with the lower left-hand-cell of λ having coordinates $(1, 1)$. The notation $x \in \lambda$ means x is a cell in λ.

Two simple functions on partitions we will often use are

$$n(\lambda) = \sum_i (i-1)\lambda_i = \sum_i \binom{\lambda'_i}{2}, \quad z_\lambda = \prod_i i^{n_i} n_i!,$$

where $n_i = n_i(\lambda)$ is the number of parts of λ equal to i.

1.4 The space of diagonal harmonics.

Let $p_{h,k}[X_n, Y_n] = \sum_{i=1}^n x_i^h y_i^k,\ h,k \in \mathbb{Z}_{\geq 0}$ denote the "polarized power sum". It is known that the set $\{p_{h,k}[X_n, Y_n],\ h,k \in \mathbb{Z}_{\geq 0}\}$ generate $\mathbb{C}[X_n, Y_n]^{S_n}$, the ring of invariants under the diagonal action. We define the quotient ring DR_n of diagonal covariants by

$$DR_n = \mathbb{C}[X_n, Y_n]/\Big\langle \sum_{i=1}^n x_i^h y_i^k, \forall h+k>0 \Big\rangle.$$

We also define the space of diagonal harmonics DH_n by

$$DH_n = \Big\{ f \in \mathbb{C}[X_n, Y_n] : \sum_{i=1}^n \frac{\partial^h}{x_i^h} \frac{\partial^k}{y_i^k} f = 0, \forall h+k>0 \Big\}.$$

The space of diagonal harmonics DH_n is a finite dimensional vector space which is isomorphic to DR_n as an S_n module. The dimension of these spaces turns out to be $(n+1)^{n-1}$ ([Hai02]).

Given a cell $x \in \lambda$, let the arm $a = a(x)$, leg $l = l(x)$, coarm $a' = a'(x)$, and coleg $l' = l'(x)$ be the number of cells strictly between x and the border of λ in the E, S, W and N directions, respectively.

For $\mu \vdash n$ define,

$$M = (1-q)(1-t), \quad B_\mu = \sum_{x\in\mu} q^{a'} t^{l'}, \quad \Pi_\mu = \prod_{x\in\mu, x\neq(1,1)} (1 - q^{a'} t^{l'})$$

$$n(\mu) = \sum_i (i-1)\mu_i, \quad T_\mu = t^{n(\mu)} q^{n(\mu')}, \quad w_\mu = \prod_{x\in\mu} (q^a - t^{l+1})(t^l - q^{a+1}).$$

Define $\tilde{K}_{\lambda,\mu}(q,t) = t^{n(\mu)} K_{\lambda,\mu}(q, 1/t)$, where $K_{\lambda,\mu}(q,t)$ are known as the q,t-Kostka polynomials. Then the "modified Macdonald polynomial" $\tilde{H}_\mu = \tilde{H}_\mu[X; q, t]$ can be defined as

$$\tilde{H}_\mu = \sum_{\lambda\vdash n} \tilde{K}_{\lambda,\mu}(q,t) s_\lambda.$$

Theorem 1.3 *(Haiman, [Hai02]).*

$$\mathcal{F}(DH_n; q, t) = \sum_{\mu\vdash n} \frac{T_\mu M \tilde{H}_\mu \Pi_\mu B_\mu}{w_\mu}.$$

1.5 Algebraic definition of q,t-Catalan numbers.

On the space of symmetric functions $\Lambda[X]$, define the Hall inner product by

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \chi(\lambda = \mu), \quad \langle s_\lambda, s_\mu \rangle = \chi(\lambda = \mu).$$

Then let

$$C_n(q,t) = \langle \mathcal{F}(DH_n; q,t), s_{1^n} \rangle = \mathcal{H}(DH_n^\varepsilon; q, t).$$

Open problem 1.4 *Find a combinatorial description of the polynomials $\langle \mathcal{F}(DH_n; q,t), s_\lambda \rangle$ for general λ.*

From Theorem 1.4 and the fact that $\langle \tilde{H}_\mu, s_{1^n} \rangle = T_\mu$, we have

$$C_n(q,t) = \sum_{\mu\vdash n} \frac{T_\mu^2 M \Pi_\mu B_\mu}{w_\mu}.$$

Garsia and Haiman ([GH96]) proved that

$$C_n(q,1) = C_n(q) = \sum_{\pi\in L_{n,n}^+} q^{area(\pi)},$$

$$q^{\binom{n}{2}} C_n(q, 1/q) = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix},$$

which shows that both the Carlitz-Riordan and MacMahon q-Catalan numbers are special cases of $C_n(q,t)$. That is why $C_n(q,t)$ is referred to as q,t-Catalan sequence.

1.6 Bounce statistic. Combinatorial description of q, t- Catalan numbers.

Given $\pi \in L^+_{n,n}$, define the *bounce path* of π to be the path described by the following algorithm. Start at $(0,0)$ and travel North along π until you encounter the beginning of an E step. Then turn East and travel straight until you hit the diagonal $y = x$. Then turn North and travel straight until you encounter again the beginning of an E step of π, then turn East and travel to the diagonal, etc. Continue until you arrive at (n, n). Let $(0,0), (j_1, j_1), (j_2, j_2), \ldots, (j_{b-1}, j_{b-1}), (j_b, j_b) = (n, n)$ are the points where the bouncing path touches the line $y = x$. Then define the bounce statistic bounce(π) to be the sum

$$\mathrm{bounce}(\pi) = \sum_{i=1}^{b-1} n - j_i.$$

Let

$$F_n(q,t) = \sum_{\pi \in L^+_{n,n}} q^{\mathrm{area}(\pi)} t^{\mathrm{bounce}(\pi)}.$$

Theorem 1.5 *(Garsia, Haglund, [GH01],[GH02])*

$$C_n(q,t) = F_n(q,t).$$

The proof of Theorem 1.6 is based on a recursive structure underlying $F_n(q,t)$. For example, it can be proved combinatorially that

$$F_n(q,t) = \sum_{i=1}^{b} \sum_{\alpha} t^{\sum_{i=2}^{b}(i-1)\alpha_i} q^{\sum_{i=1}^{b}\binom{\alpha_i}{2}} \prod_{i=1}^{b-1} \begin{bmatrix} \alpha_i + \alpha_{i+1} - 1 \\ \alpha_{i+1} \end{bmatrix},$$

where the inner sum is over all compositions α of n into b positive integers.

1.7 The symmetry problem and the dinv statistic.

From it's algebraic definition it's easy to show $C_n(q,t) = C_n(t,q)$. Thus we have

Corollary 1.6

$$\sum_{\pi \in L^+_{n,n}} q^{\mathrm{area}(\pi)} t^{\mathrm{bounce}(\pi)} = \sum_{\pi \in L^+_{n,n}} q^{\mathrm{bounce}(\pi)} t^{\mathrm{area}(\pi)}.$$

At present there is no other known way to prove this equality other than as a corollary of Theorem 1.6.

Open problem 1.7 *Prove Corollary 1.7 by exhibiting a bijection on Dyck paths which interchanges area and bounce.*

There is another pair of statistics for the q, t-Catalan discovered by M.Haiman. It involves pairing area with a different statistic called dinv, for "diagonal inversion" or "d-inversion". It is defined, with a_i the length of the ith row from the bottom, as follows. For $\pi \in L^+_{n,n}$, let

$$\mathrm{dinv}(\pi) = |\{(i,j) : 1 \le i < j \le n \quad a_i = a_j\}| + |\{(i,j) : 1 \le i < j \le n \quad a_i = a_j + 1\}|.$$

Or, equivalently, let $\lambda(\pi)$ denote the partition above π but inside the $n \times n$ square. Then

$$\mathrm{dinv}(\pi) = |\{s \in \lambda(\pi) : \mathrm{leg}(s) \le \mathrm{arm}(s) \le \mathrm{leg}(s) + 1\}|.$$

Theorem 1.8

$$\sum_{\pi \in L^+_{n,n}} q^{\mathrm{dinv}(\pi)} t^{\mathrm{area}(\pi)} = \sum_{\pi \in L^+_{n,n}} q^{\mathrm{area}(\pi)} t^{\mathrm{bounce}(\pi)}.$$

There's a combinatorial proof of Theorem 1.9 that describes a bijective map $\zeta : L^+_{n,n} \to L^+_{n,n}$ such that

$$\mathrm{dinv}(\pi) = \mathrm{area}(\zeta(\pi)), \quad \mathrm{area}(\pi) = \mathrm{bounce}(\zeta(\pi)).$$

1.8 Rational Dyck paths.

Define the hook length of the cell $x \in \lambda$ as $\mathrm{hl}(x) = \mathrm{arm}(x) + \mathrm{leg}(x) + 1$. An (a, b)-core is a partition λ such that for any $x \in \lambda$, the hook length of x is not equal to a or b. We define the set of (a, b)-cores as $\mathcal{C}_{a,b}$.

Suppose $(a, b) = 1$. Then there is a bijection between (a, b)-cores and rational Dyck paths from $L^+_{a,b}$ called Anderson's bijection.

The *hook filling* of the boxes in the square lattice is obtained by filling the box with lower-right lattice point $(b, 0)$ with the number $-ab$ and increasing by a for every one box west and increasing by b for every one box north. A box is above the main diagonal if and only if the corresponding hook is positive. The *positive hooks* of $\pi \in L^+_{a,b}$ are the numbers in the hook filling below the path but greater than zero. The number of positive hooks is exactly the area of π. We denote $c(\pi)$ the (a, b)-core corresponding to π under Anderson bijection: the hook lengths of the boxes in the first column of $c(\pi)$, its *leading hooks*, are precisely the positive hooks of π.

It's often easier to work with (a, b)-cores instead of rational Dyck paths.

Let κ be an a-core partition. Consider the hook lengths of the boxes in the first column of κ. Find the largest hook length of each residue modulo a. The *a-rows* of κ are the rows corresponding to these hook lengths. The *a-boundary* of κ consists of all boxes in it's Young diagram with hook length less than a.

Let κ be an (a, b)-core partition. The *skew length* of κ, denoted $\mathrm{sl}(\kappa)$, is the number of boxes simultaneously located in the a-rows and the b-boundary of κ.

An interesting property of $\mathrm{sl}(\kappa)$ is that it is independent of the ordering of a and b ([CDH15]).

The *co-skew length* of an (a, b)-core κ is

$$\mathrm{sl}'(\kappa) = \frac{(a-1)(b-1)}{2} - \mathrm{sl}(\kappa).$$

The *rank* of π, denoted $\mathrm{rk}(\pi)$ is the number of rows in $\lambda(\pi)$.

The analogue of the dinv statistic on rational Dyck paths can be defined as

$$\mathrm{dinv}(\pi) = \left|\left\{ s \in \lambda(\pi) : \frac{\mathrm{arm}(s)}{\mathrm{leg}(s)+1} \le \frac{b}{a} < \frac{\mathrm{arm}(s)+1}{\mathrm{leg}(s)} \right\}\right|.$$

Open problem 1.9 *Find an analogue of the bounce statistic on rational Dyck paths.*

Conjecture 1.10 *Let a and b relatively prime positive integers. Then*

$$\frac{1}{[a+b]}\begin{bmatrix} a+b \\ a \end{bmatrix} = \sum_{\kappa \in \mathcal{C}_{a,b}} q^{\mathrm{sl}(\kappa)+\mathrm{rk}(\kappa)}.$$

Define the rational q, t-Catalan numbers as

$$F_{a,b}(q,t) = \sum_{\kappa \in \mathcal{C}_{a,b}} q^{\mathrm{rk}(\kappa)} t^{\mathrm{sl}'(\kappa)}.$$

Conjecture 1.11

$$\sum_{\kappa \in \mathcal{C}_{a,b}} q^{\mathrm{rk}(\kappa)} t^{\mathrm{sl}'(\kappa)} = \sum_{\kappa \in \mathcal{C}_{a,b}} q^{\mathrm{sl}'(\kappa)} t^{\mathrm{rk}(\kappa)}.$$

1.9 ζ- map on rational Dyck paths.

For $\pi \in L^+_{a,b}$, let $\nu(\pi) = (\nu_1, \ldots, \nu_a)$ be the partition that has parts equal to the number of b-boundary boxes in the a-rows of $c(\pi)$.

Define $\zeta(\pi)$ to be the (a, b)-Dyck path such that $\lambda(\zeta(\pi)) = \nu(\pi)$.

Proposition 1.12

$$\mathrm{sl}'(\pi) = \mathrm{area}(\zeta(\pi)), \quad \mathrm{dinv}(\pi) = \mathrm{area}(\zeta(\pi)).$$

Corollary 1.13

$$\mathrm{sl}'(\pi) = \mathrm{dinv}(\pi).$$

Open problem 1.14 *Prove that ζ is bijective.*

2 Topics (with references)

1. Combinatorics.

   - Enumerative Combinatorics. ([EC1], [AKS13])
     - Cycles and inversions. Descents. Partitions and q-binomial coefficients. Partition identities. The twelvefold way.
     - Inclusion-exclusion formula. Permutations with restricted position. Involutions.
     - Posets. Lattices. Distributive lattices. Incidence algebras. Moebius inversion formula. Promotion and evacuation.
     - Markov chains on linear extensions.
   - Symmetric Functions. ([EC2], Ch.7)
     - Monomial, elementary, complete homogeneous, power sum symmetric functions. An involution. A scalar product.
     - Schur functions: combinatorial definition, classical definition.
     - Semi-standard Young tableaux. The RSK algorithm.
     - The Jacobi-Trudi identity. The Murnaghan-Nakayama rule. The Littlewood-Richardson Rule.
   - Algebraic Combinatorics. ([EC2], [Hag08])
     - The characters of the Symmetric group.
     - Hilbert series, Frobenius series.
     - Macdonald Polynomials and the Space of Diagonal harmonics.
   - q, t- Catalan numbers. ([Hag08], Ch.2,3)
     - Statistics on Dyck paths: Bounce statistic, Dinv statistic. The Zeta map on rational Dyck paths.
     - Definition of q, t- Catalan numbers.
     - Special values $t = 1$ and $t = 1/q$.
     - The Symmetry Problem.
   - Enumeration of Integer Points in Polyhedra. ([Ba08])
     - The algebra of polyhedra. Linear transformations. Polarity. Tangent cones and decompositions modulo polyhedra with lines. Open polyhedra.
     - The exponential valuation. Lattices, bases and parallelepipeds. The Minkowski Convex Body theorem.
     - Exponential sums and generating functions. Totally unimodular polytopes. Decomposing a rational cone into unimodular cones. Efficient counting of integer points in rational polytopes.
     - The polynomial behavior of the number of integer points in polytopes. A valuation on rational cones.

2. Algebra and Representation Theory. ([DF04], [Bur65])

   - Introduction to Group Theory. ([DF04], Ch.1-6)
     - Basic Groups. Subgroups. Quotient groups and Homomorphisms.
     - Group actions. Sylow's theorem.
     - Direct and semidirect products. The fundamental theorem of finitely generated Abelian groups.
     - p-Groups, nilpotent groups and solvable groups.
   - Representation Theory of Rings with Identity. ([DF04], [Bur65])

    - Rings, Modules, Vector Spaces. Ring homomorphisms, quotient rings and ideals. Module homomorphisms and quotient modules. The matrix of linear transformation. Direct sum.
    - Representation modules. The regular representation.
    - The principle indecomposable representations. The radical of a ring. Semisimple rings. The Wedderburn structure theorems for semisimple rings.
    - Intertwining numbers. Multiplicities of the indecomposable components in the regular representation.
  - Representation Theory of Finite Groups and Theory of Characters. ([Bur65])
    - The group algebra. Semisimplicity of the group algebra. The center of the group algebra. The number of inequivalent irreducible representations.
    - Orthogonal relations on the irreducible characters of the group. Module of characters over the integers, symmetric bilinear form on characters.
    - The Kronecker product of two representations. Induced representations and induced characters.
    - Normal subgroups and the character table. Representations of cyclic groups and abelian groups.

3. Complex Analysis. ([SS03])
   - Cauchy's Theorem and Applications.
   - Meromorphic Functions and the Logarithm.
   - The Fourier Transform.
   - Entire Functions.
   - The Gamma and Zeta Functions.

4. Probability Theory. ([Dur05])
   - Laws of Large Numbers.
   - Central Limit Theorems.
   - Random Walks.
   - Martingales.
   - Markov Chains.
   - Brownian Motion.

5. Real Analysis. ([HN01])
   - Banach Spaces.
   - Hilbert Spaces.
   - Fourier Series.
   - Distributions and the Fourier Transform.
   - Measure Theory and Function Spaces.

3 References

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