Analysis Preliminary Exams Solutions Guide

UC Davis Department of Mathematics

The Galois Group

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Chapter 1

Building Up to the Exam

1.1 Your Graduate Education in Mathematics

Welcome to the Department of Mathematics at UC Davis. You have been chosen as one of the elite individuals who will contribute to future mathematical knowledge as a member of our department. You are bright, capable and incredibly intelligent. We hope that you will reach for the stars and find great success over the coming years. The following mannual is written for graduate students by graduate students. It is designed to help you navigate the annals of Mathematical Analysis offered during your first 365 days in our Department. The goal of this mannual is to aid you in passing the Analysis Preliminary Exam with flying colors before the start of your second year. Use this manual as one of the many tools you have available here at UC Davis to learn Analysis. Your predicessors wish you luck and know that you will do spectacularly as you embark on your Graduate Education in Mathematics here at UC Davis.

As is discussed in the program brochures, the graduate programs in Pure and Applied Mathematics are loosely divided into a series of milestones including

(a) passing the preliminary examination

(b) finishing your course work

(c) passing the qualifying examination

(d) writing your thesis

The department and faculty take responsibility to guide our Graduate Students through each of these. However, no member of this department is more prepared to help you achieve your goals and find success here than yourself. As a mathematics graduate student you might consider dedicating yourself to this endevour and make this part of your everyday life. The assumptions

> "If I show up to class and do the homework, I will learn the material."

and

> "My teacher will teach me everything I need to know."

are as valid as the statement

$$1 + 1 = 3$$

In your first year here at Davis, you will be required to take the Math 201 series which is comprised of the following three courses:

(a) Math 201A

(b) Math 201B

(c) Math 201C

Generally, these courses introduce the students to the topics in Analysis that the department faculty feels we need (Steve Skholler, John Hunter, Becca Thomases and others who contribute to syllabus creation). The ironic thing about most of these courses is that, by themselves will not teach you math. To learn math is your job! Go beyond the classroom. Find ways to be resourceful and to exceed the requirements of each course you take. Do not assume that the one book you might be reading for this class is sufficient. Do not assume that the leader of your course is capable of teaching you what you need to know. Think instead of the Faculty leading this course as a tour guide who is showing you some of the scenary. It is your job to get out of the bus, get out your magnifying glass and get dirty as you explore the environment. The Preliminary exams ask you to demonstrate that you have done exactly this.

1.2 Exam Specifications

The Preliminary Exam in Analysis covers the following topics.

(a) Continuous function: Convergence of functions, Spaces of Continuous functions; Approximations by Polynomials; Arezela Ascoli theorem

(b) Banach Spaces: Bounded linear operators; Different notions of BLO convergence; Compact Operators; Dual Spaces; Finite Dimensional Banach Spaces

(c) Hilbert Spaces: Orthogonalizty; Orthonormal bases; Parseval's identity

(d) Fourier series: Convolution; Young's Inequality; Fourier Series of differentiable functions; Sobolev Embedding Theorem

(e) Bounded Linear Operators on a Hilbert Space: Orthogonal projections; The dual of a Hilbert Space (Riesz representation); The adjiont of an operator; Self-adjoint and unitary operators; Weak convergence theorem; Hilbert-Schmidt operators; Functions of operators

(f) The spectral theory for compact, self adjoint operators: Spectrum; Compact operators; the spectral theorem; Hilbert-Schmidt operators; Functions of operators

(g) Weak derivatives

(h) Fourier Transfrom: The Fourier Transform on L_1 and L_2; The Poisson summation formula.

It is designed by the faculty here at UC Davis as a Rite of Passage for our graduate students. Students who study and pass the preliminary exam have accomplished their first major task as a graduate math student in our department. // The road to success in this exam is worth a year of devout study. The entire purpose of your first year of course work is to get you up to par with this material by introducing formal mathematical arguments in a

1.3 Using These Solutions

Chapter 2

References and Study Aids

2.1 Reference Texts in Analysis

A major difference between undergraduate and graduate education lies in the expectations of the student. As graduate students, we are expected to have deft command of topics of analysis from the basics of series and sequences through weak convergences and Sobelev spaces. The classic assumption of the student

> If I read the book, show up to lecture and do the homework, I can adequately master the material

is hardly sufficient for graduate education. We must push ourselves to dive deeper. We must find the energy and support to train ourselves in not only problem solving skills but breadth and depth of knowledge. Passing your preliminary exam indicates that you have achieved the minimal level of mastery required by the department. However, take the first year of your course work to go beyond the call of duty. Find a way to reach into the depths of this material and learn more about the intricate details. Your hard work WILL pay off, both on this exam and in your future mathematical endeavors.

Searching for deeper understanding in mathematics is a life time pursuit. The professors in the department have made a living through this search. Take a look at the book shelfs of your favorite math professor. How many books do you see? With a probability that tends to one as your sample size increases, you see a tens if not hundreds of mathematics reference books and text books. These range from introductory to quite specfic. Consider making your journey through analysis by emulating this trend. Get resources and references that are going to help you succeed both in your course work and on these exams. Do not assume that the texts required in the 201 series are sufficient for your needs. Budget, borrow, sample and buy mathematical texts which you can use to your advantage. In the following subsections you will find a small advertisement about references to help you on your journey. Each of these books has been used by previous graduate students here at UC Davis. They are highly recommended by past graduate students as study tools. Consider each of these as an investment that you can keep in your professional library.

2.1.1 Berkeley Problems in Mathematics by Souza and Silva

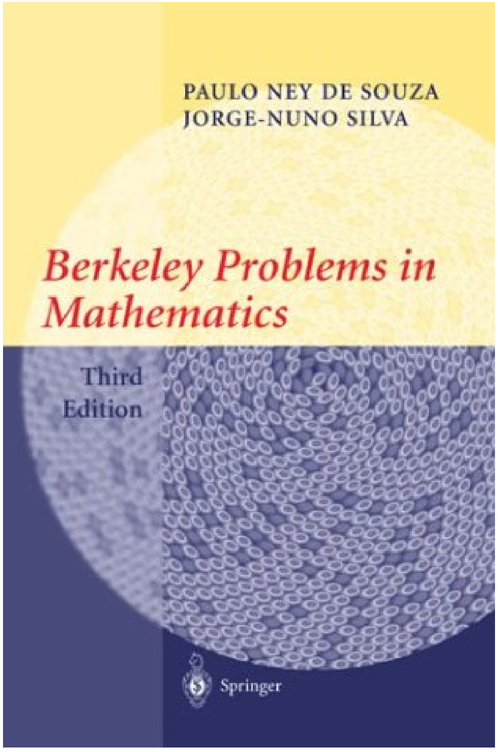

Figure 2.1: ISBN: 0387008926, Price: $34.95, Shields Library QA43 .D38 2001 Regular Loan

One of the most important skills in passing the preliminary exam is problem solving in a fast and efficient manner. You must be able to recognize the heart of the problem, recall relevant theorems and manipulate the antecedents to arrive at the conclusion. All this in about 30 minutes. This book is designed to help you become a better problem solver. Here is an Amazon.com review:

"This book collects approximately nine hundred problems that have appeared on the preliminary exams in Berkeley over the last twenty years. It is an invaluable source of problems and solutions. Readers who work through this book will develop problem solving skills in such areas as real analysis, multivariable calculus, differential equations, metric spaces, complex analysis, algebra, and linear algebra."

"The Mathematics department of the University of California, Berkeley, has set a written preliminary examination to determine whether first year Ph.D. students have mastered enough basic mathematics to succeed in the doctoral program. Berkeley Problems in Mathematics is a compilation of all the É questions, together with worked solutions É . All the solutions I looked at are complete É . Some of the solutions are very elegant. É This is an impressive piece of work and a welcome addition to any mathematicianÕs bookshelf." (Chris Good, The Mathematical Gazette, 90:518, 2006)

2.1.2 Foundations of Mathematical Analysis by Johnsonbaugh and Pfaffenberger

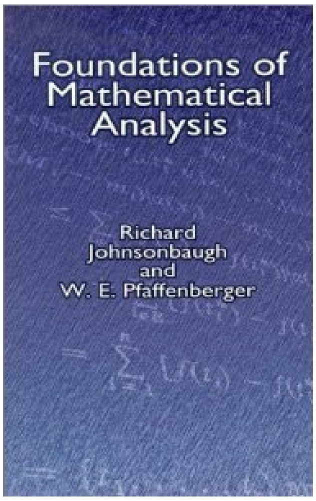

Figure 2.2: ISBN: 0486421740, Price: \$22.95, Shields Library QA299.8 .J63

Foundations of Mathematical Analysis by Richard Johnsonbaugh and W.E. Pfaffenberger is a great introduction to real analysis. Topics range from the axiomatic definition of the real numbers through The Riesz Representation Theorem and Lebesgue's Intergral. For students who want to have a concise listing of the foundations of analysis and a wide range of accessible practice problems for extra support in Analysis, this is a good reference to have. Solutions to earlier preliminary exam questions can be found in this text. For example

- Winter 2002 Problem 1 (pg 282)
- Winter 2002 Problem 2 (pg 247)
- Winter 2005 Problem 3 (pg 249)

From the preface

> This book evolved from a one-year Advanced Calculus course that we have given during the last decade. Our audiences have included junior and senior majors and honors students, and on occasion, gifted sophomores... Our intent is to teach students the tools of modern analysis as it relates to further study in mathematics, especially statistics, numerical analysis, differential equations, mathematical analysis and functional analysis.
>
> Because we believe that an essential part of learning mathematics is doing mathematics, we have included over 750 exercises, some containing several parts, of varying degree of difficulty. Hints and solutions to selected exercises are given at the back of the book.

2.1.3 Principles of Mathematical Analysis by Rudin

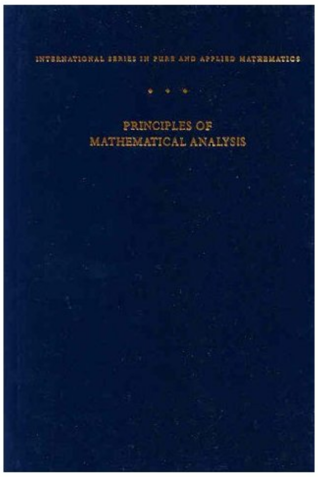

Figure 2.3: ISBN: 007054235X, Shields Reserves Reserves QA300 .R8 1976

Principles of Mathematical Analysis by Walter Rudin is a classic text in this subject. Similar to the foundations of mathematical analysis above, Rudin takes his readers on a tour of topics ranging from the axiomatic approach to the Real and complex numbers through Function spaces and Lebesgues measure. This book is particularly strong in its approach to function spaces and uniform continuity. It can be used to establish the intuition for L^p spaces, which are fundamental in understanding Sobolev spaces and weak convergences.

2.1.4 Fourier Analysis by Kammler

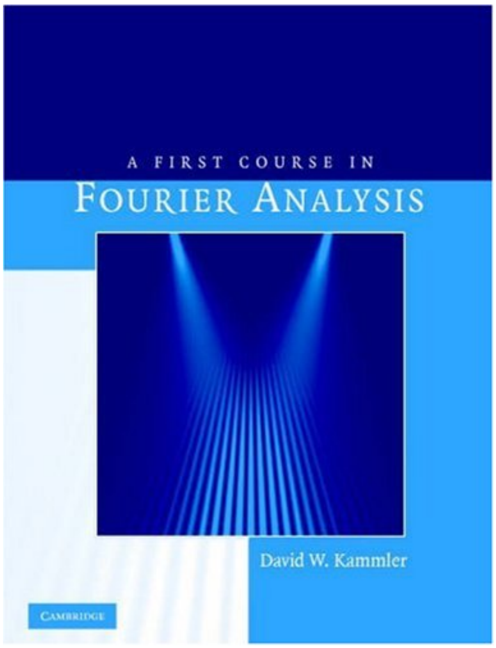

Figure 2.4: ISBN: 0521709792, Price: $75.00, Shields Library QA403.5 .K36 2007

From the Preface:

> This unique book provides a meaningful resources for applied mathematics through Fourier analysis. It develops a unified theory of discrete and continuous (univariate) Fourier analysis, the fast Fourier transform, and a powerful elementary theory of generalized functions, including the use of weak limits. It then shows how these mathematical ideas can be used to expedite the study of sampling theory, PDEs, wavelets, probability, diffraction, etc. Unique features include a unified development of Fourier synthesis/analysis for functions on $\mathbb{R}, \mathbb{T}_p, \mathbb{Z}$ and $\mathbb{P}_N$; an unusually complete development of the Fourier transform calculus (for finding Fourier transforms, Fourier series, and DFTs); memorable derivations of the FFT; a balanced treatment of generalized functions that fosters mathematical understanding as well as practical working skills; a careful introduction to Shannon's sampling theorem and modern variations; a study of the wave equation, diffusion equation, and diffraction equation by using the Fourier transform calculus, generalized functions and weak limits; an exceptionally efficient development of Daubechies' compactly supported orthogonal wavelets;... A valuable reference of Fourier analysis for a variety of scientific professionals, including Mathematicians, Physicists, Chemists, Geologists, Electrical Engineers, Mechanical Engineers and others.

2.1.5 Real and Complex Analysis by Rudin

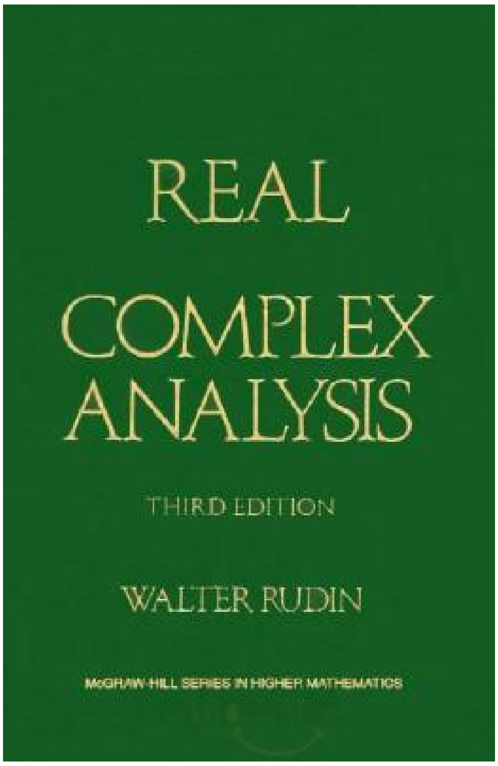

Figure 2.5: ISBN: 0070542341, Price: $98.99, Shields Library QA300 .R82 1987

Product Review (Amazon.com):

> The first part of this book is a very solid treatment of introductory graduate-level real analysis, covering measure theory, Banach and Hilbert spaces, and Fourier transforms. The second half, equally strong but often more innovative, is a detailed study of single-variable complex analysis, starting with the most basic properties of analytic functions and culminating with chapters on Hp spaces and holomorphic Fourier transforms. What makes this book unique is Rudin's use of 20th-century real analysis in his exposition of "classical" complex analysis; for example, he uses the Hahn-Banach and Riesz Representation theorems in his proof of Runge's theorem on approximation by rational functions. At times, the relationship circles back; for example, he combines work on zeroes of holomorphic functions with measure theory to prove a generalization of the Weierstrass approximation theorem which gives a simple necessary and sufficient condition for a subset S of the natural numbers to have the property that the span of $\{t^n : ninS\}$ is dense in the space of continuous functions on the interval. Real and Complex Analysis is at times a fascinating journey through the relationships between the branches of analysis.

2.1.6 Real Analysis by Folland

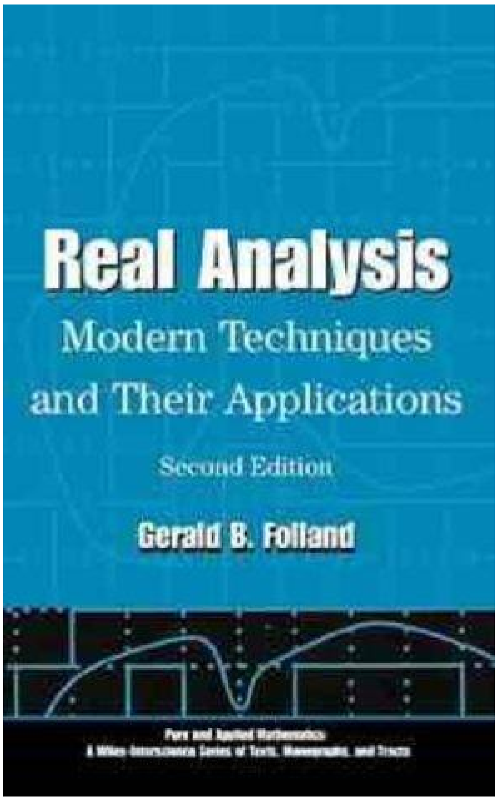

Figure 2.6: ISBN: 0471317160, Price: \$129.99, Shields Library QA300 .F67 1999

From the Preface

> The name "real analysis" is something of an anachronism. Originally applied to the theory of functions of a real variable, it has come to encompass several subject of a more general and abstract nature that underlie much of modern analysis. These general theories and their applications are the subject of this book, which is intended primarily as a text for a graduate-level analysis course. Chapters 1 through 7 are devoted to the core material from measure and integration theory, point set topology, and functional analysis that is part of most graduate curricula in mathematics, together with a few related but less standard items with which I think all analysts should be acquainted. The last four chapters contain a variety of topics that are meant to introduce some of the other branches of analysis and to illustrate the uses of the preceding material. I believe these topics are all interesting and important, but their selection in preference to other is largely a matter or personal predilection.

2.1.7 Analysis by Lieb and Loss

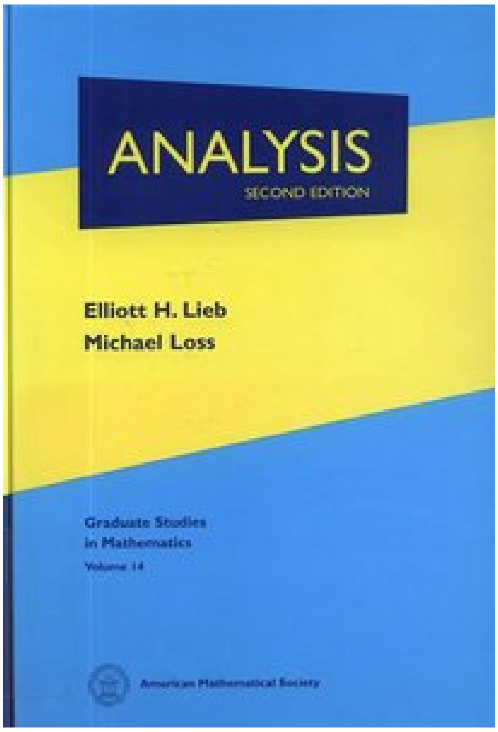

Figure 2.7: ISBN: 0821827839, Price:$35.00, Shields Library QA300.L54 2001

From the preface

> Originally, we were motivated to present the essentials of modern analysis to physicists and other natural scientists, so that some modern developments in quantum mechanics, for example, would be understandable. From personal experience we realize that this task is a little different form the task of explaining analysis to students of mathematics... Throughout, our approach is 'hands on', meaning that we try to be as direct as possible and do not always strive for the most general formulation. Occasionally we have slick proofs, but we avoid unnecessary abstraction, such as the use of the Baire category theorem or the Hahn-Banach theorem, which are not needed for L^p spaces. Our preference is to understand L^p-spaces and then have the reader go elsewhere to study Banach spaces generally, rather than the other way around. Another noteworthy point is that we try not to say, 'there exists a constant such that...'. We usually give it, or at least an estimate of it. It is important for students of the natural sciences and mathematics, to learn how to calculate. Nowadays, this is often overlooked in mathematics courses that usually emphasize pure existence theorems.

2.1.8 Applied Analysis by Hunter and Nachtergaele

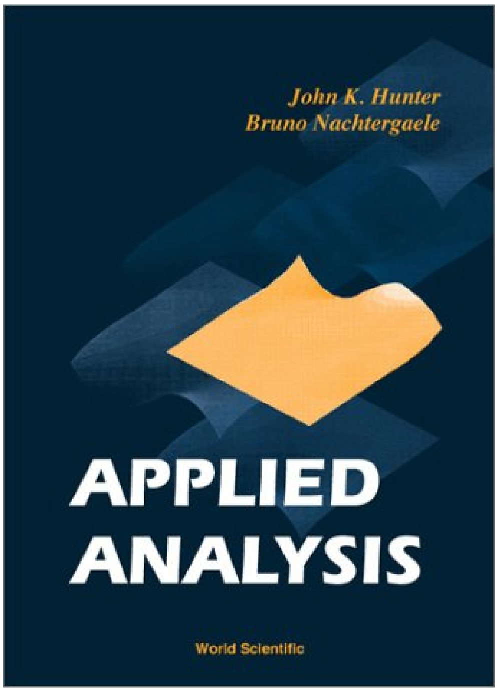

Figure 2.8: ISBN: 9810241917, Price: $92.00, Shields Library QA300 .H93 2001

From the Preface

> The aim of this book is to supply an introduction for beginning graduate students to those parts of analysis that are most useful in applications. The material is selected for its use in applied problems, and is presented as clearly and simply as we are able, but without the sacrifice of mathematical rigor...
>
> We provide detailed proofs for the main topics. We make no attempt to state results in maximum generality, but instead illustrate the main ideas in simple, concrete settings. We often return to the same ideas in different contexts, even if this leads to some repetition of previous definitions and results. We make extensive use of examples and exercises to illustrate the concepts introduced. The exercises are at various levels; some are elementary, although we have omitted many of the routine exercises that we assign while teaching the class, and some are harder and are an excuse to introduce new ideas or application not covered in the main text. One area where we do not give a complete treatment is Lebesgue measure and integration. A full development of measure theory would take us too far afield, and in any event, the Lebesgue integral is much easier to use than to construct.

2.1.9 Measure Theory by Cohn

Figure 2.9: ISBN: 0817630031, Price: $55.10, Shields Library QA312 .C56

Product Review (Amazon.com)

> Intended as a straightforward introduction to measure theory, this textbook emphasizes those topics relevant and necessary to the study of analysis and probability theory. The first five chapters deal with abstract measure and integration. At the end of these chapters, the reader will appreciate the elements of integration. Chapter 6, on differentiation, includes a treatment of changes of variables in Rd. A unique feature of the book is the introductory, yet comprehensive treatment of integration on locally Hausdorff spaces, of the analytic and Borel subsets of Polish spaces, and of Haar measures on locally compact groups. Measure Theory provides the reader with tools needed for study in several areas of current interest, in particular harmonic analysis and probability theory, and is a valuable reference tool.

This text is a beautiful introduction to measure theory and should be used to deepen the readers understanding of Lieb and Loss chapter 1 and Hunter and Natergaele Chapter 12. It comes in very useful

2.1.10 Functional Analysis by Reed and Simon

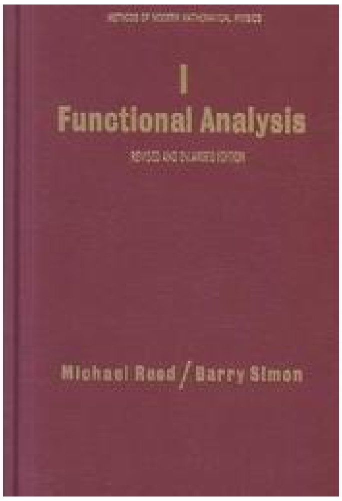

Figure 2.10: ISBN: 0125850506, Price: $123.21, Phy Sci Engr Library QC20.7.F84 R43 1980 Regular Loan

Product Review (Amazon.com)

> This book is the first of a multivolume series devoted to an exposition of functional analysis methods in modern mathematical physics. It describes the fundamental principles of functional analysis and is essentially self-contained, although there are occasional references to later volumes. We have included a few applications when we thought that they would provide motivation for the reader. Later volumes describe various advanced topics in functional analysis and give numerous applications in classical physics, modern physics, and partial differential equations.

Chapter 3

Fall 2002

Author: Luke Grecki

3.0.11 Problem 6

Problem 6

Let $f_n : \mathbb{R} \to \mathbb{R}$ be a differentiable mapping for each n, with $|f_n^{'}(x)| \leq 1$ for all n, x. Show that if $g(x)$ is a function such that

$$\lim_{x\to\infty} f_n(x) = g(x)$$

then $g(x)$ is a continuous function.

Proof:
Let $x_0 \in \mathbb{R}$. By the mean value theorem and the bound on $f_n^{'}$ we see

$$|f_n(x) - f_n(x_0)| \leq \sup_x |f_n^{'}(x)| \cdot |x - x_0| \leq |x - x_0|$$

Now let $\epsilon > 0$ and $\delta = \frac{\epsilon}{3}$. For any x satisfying $|x - x_0| \leq \delta$ we know $|f_n(x) - f_n(x_0)| \leq \frac{\epsilon}{3}$ from the above. The pointwise convergence of f_n to g implies that there exist $N \in \mathbb{N}$ such that

$$|f_n(x) - g(x)| \leq \frac{\epsilon}{3}$$

$$|f_n(x_0) - g(x_0)| \leq \frac{\epsilon}{3}$$

for all $n \geq N$. By adding and subtracting terms and using the triangle inequality we find

$$|g(x) - g(x_0)| \leq |g(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - g(x_0)| \leq \epsilon$$

which shows that g is continuous. $\square$

3.0.12 Problem 7

Problem 7

(a) State the Stone-Weierstrass theorem in the context of $C(X, \mathbb{R})$, where X is a compact Hausdorff space.

(b) State the Radon-Nikodym theorem, as it applis to a pair of σ-finite measures μ and ν defined on a measurable space $(X, \mathcal{M})$.

(c) State the definitions of the terms *normal* topological space and *absolutely continuous function*.

Proof:
Obtained by opening the right book and turning to the right page.

□

3.0.13 Problem 8

Problem 8

Let $f : X \to Y$ be a mapping between topological spaces X and Y. Let $\mathcal{E}$ be a base for the topology of Y. Show that if $f^{-1}(E)$ is open in X for each $E \in \mathcal{E}$, then f is continuous.

Proof:
Let $V \subset Y$ be open. We must show that $f^{-1}(V) \subset X$ is open. Since $\mathcal{E}$ is a base for the topology of Y we can write

$$V = \bigcup_{\alpha} E_\alpha$$

for some sets $E_\alpha \in \mathcal{E}$. Since unions are preserved under inverse images we have

$$f^{-1}(V) = f^{-1}\left(\bigcup_{\alpha} E_\alpha\right) = \bigcup_{\alpha} f^{-1}(E_\alpha)$$

By hypothesis every $f^{-1}(E_\alpha)$ is open, and since Y is a topological space their union must be open. Therefore $f^{-1}(V)$ is open and f is continuous. $\square$

3.0.14 Problem 9

Problem 9

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing and right-continuous function, and let μ be the associated measure, so that $\mu(a, b] = F(b) - F(a)$ for $a < b$. Prove that $\mu(\{a\}) = F(a) - F(a^-)$ and $\mu[a, b] = F(b) - F(a^-)$ for all $a < b$. [Notation: $F(a^-) := \lim_{x\to a^-} F(x)$].

Proof:
To show the former consider the sequence of nested sets

$$(a-1, a] \supset (a - \frac{1}{2}, a] \supset (a - \frac{1}{3}, a] \supset \cdots$$

Since μ is a measure and $\mu(a - 1, a] < \infty$ we have

$$\begin{aligned}\mu(\{a\}) &= \mu\left(\bigcap_{n=1}^{\infty} (a - \frac{1}{n}, a]\right) \\ &= \lim_{n\to\infty} \mu(a - \frac{1}{n}, a] \\ &= \lim_{n\to\infty} F(a) - F(a - \frac{1}{n}) \\ &= F(a) - F(a^-)\end{aligned}$$

which is what we wanted. Note that the latter limit $F(a^-)$ exists since F is increasing and $F(a) < \infty$.

To show the latter we observe that $[a, b] = \{a\} \bigsqcup (a, b]$. Using the additivity of μ we get

$$\begin{aligned}\mu[a, b] &= \mu(\{a\}) + \mu(a, b] \\ &= F(b) - F(a^-)\end{aligned}$$

□

3.0.15 Problem 10

Problem 10

Let f be a $\mathcal{B}_{[0,1]^2}$-measurable real-valued function such that the partial derivative $\dfrac{\partial f}{\partial t}(x,t)$ exists for each $(x,t) \in [0,1]^2$ and $M := \sup \left|\dfrac{\partial f}{\partial t}(x,t)\right| < \infty$. Prove that $\dfrac{\partial f}{\partial t}(x,t)$ is measurable, and that for all $t \in [0,1]$,

$$\frac{d}{dt}\int_0^1 f(x,t)\,dx = \int_0^1 \frac{\partial f}{\partial t}(x,t)\,dx$$

Proof:
First we show that $\dfrac{\partial f}{\partial t}(x,t)$ is measurable as a function of x. By definition,

$$\frac{\partial f}{\partial t}(x,t) = \lim_{t'\to t} \frac{f(x,t') - f(x,t)}{t'-t}$$

Since f is $\mathcal{B}_{[0,1]^2}$-measurable its restriction $f_t(x) = f(x,t)$ is $\mathcal{B}_{[0,1]}$-measurable. The sum and product of measurable functions is a measurable function so the quotients

$$q_{t'}(x) = \frac{f(x,t') - f(x,t)}{t'-t} = \frac{f'_t(x) - f_t(x)}{t'-t}$$

are measurable. Furthermore the pointwise limit of measurable functions is measurable so

$$\frac{\partial f}{\partial t}(x,t) = \lim_{t'\to t} q_{t'}(x)$$

is $\mathcal{B}_{[0,1]}$-measurable as a function of x. To show the equality we first rewrite the left hand side as a limit

$$\frac{d}{dt}\int_0^1 f(x,t)\,dx = \lim_{n\to\infty}\int_0^1 n \times \left[f\left(x, t+\frac{1}{n}\right) - f(x,t)\right] dx$$

By the mean value theorem we have

$$\left| n \times \left[f\left(x, t+\frac{1}{n}\right) - f(x,t)\right]\right| \leq \left| n \times \left[\frac{\partial f}{\partial t}(x,t') \times \frac{1}{n}\right]\right| \leq M$$

where $t < t' < t + \frac{1}{n}$. The last inequality follows from the hypothesized bound on $\dfrac{\partial f}{\partial t}$. Note that the constant function $g(x) \equiv M$ is in $L^1[0,1]$. By applying Lebesgue's dominated convergence theorem and using the mean value theorem again we conclude

$$\begin{aligned}\frac{d}{dt}\int_0^1 f(x,t)\,dx &= \int_0^1 \lim_{n\to\infty} n \times \left[f\left(x, t+\frac{1}{n}\right) - f(x,t)\right] dx \\ &= \int_0^1 \frac{\partial f}{\partial t}(x,t)\,dx\end{aligned}$$

□

3.0.16 Problem 11

Problem 11

Let m denote the Lebesgue measure on $\mathbb{R}$, fix $f \in L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$, and define a function $G : \mathbb{R} \to \mathbb{R}$ by the formula

$$G(t) = \int_{\mathbb{R}} f(x+t)\, dm(x)$$

Prove that G is a continuous function.

Proof:
First note that since $f \in L^1$ the integral

$$\int_{\mathbb{R}} f(x)\, dm(x)$$

is well-defined and finite. The Lebesgue measure m is translation invariant, which implies that

$$\int_{\mathbb{R}} f(x+t)\, dm(x) = \int_{\mathbb{R}} f(x)\, dm(x)$$

for all $t \in \mathbb{R}$. Therefore $G(t)$ is constant, and thus continuous.

$\square$

Chapter 4

Winter 2002

Author: Adam Sorkin

4.0.17 Problem 1

Statement: Problem Number 1

Show that the ℓ^2 norm is indeed a norm.

Proof:
Recall that $\ell^2(\mathbb{N})$ is the space of all square summable sequences of complex numbers. Letting a_i denote one such sequence, let us temporarily denote

$$f(a_i) = \sqrt{\sum_i |a_i|^2}.$$

To show that f is indeed a norm, we must have that f is positive definite, homogeneous, and satisfies the triangle inequality. Positive definiteness follows immediately from the definition, and similarly homogeneity. As usual, only the third takes any real work. To prove that $f(a_i + b_i) \le f(a_i) + f(b_i)$, we begin with the simple observation that

$$0 \le (|a_i b_j| - |a_j b_i|)^2.$$

Expanding this out and summing over $i \ne j$ gives

$$\sum_{i \ne j} |a_i a_j b_i b_j| \le \sum_{i \ne j} |a_i|^2 |b_j|^2$$

If we add to both sides the quantity $\sum_i |a_i b_i|^2$ and rearrange slightly, we get

$$\left(\sum_i |a_i b_i|\right)^2 \le \left(\sum_i |a_i|^2\right) \cdot \left(\sum_j |b_j|^2\right) = f(a_i)^2 f(b_i)^2.$$

Taking roots gives $\sum_i |a_i b_i| \leq f(a_i)f(b_i)$, and using the triangle inequality on complex numbers gives

$$\begin{aligned} f(a_i + b_i)^2 &= \sum_i |a_i + b_i|^2 \\ &\leq \sum_i \left(|a_i|^2 + |b_i|^2 + 2|a_i b_i|\right) \\ &\leq f(a_i)^2 + f(b_i)^2 + 2f(a_i)f(b_i) \\ &\leq (f(a_i) + f(b_i))^2 \end{aligned}$$

Taking roots gives the desired inequality. □

4.0.18 Problem 2

Statement: Problem Number 2

> Prove that $C([0,1])$, the space of continuous functions on $[0,1]$, is not complete in the L^1 metric: $\rho(f,g) = \int |f(x) - g(x)|dx$.

Proof:
Consider the sequence of functions $f_n \colon [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$. Clearly these functions are continuous and integrable, hence in $C([0,1])$. Moreover, f_n is Cauchy, for we can compute $\|f_n\| = 1/(n+1)$.
Now f_n has no limit in $C([0,1])$, for any L^1-limit of f_n must be a pointwise limit of f_n. But f_n converges pointwise to a discontinuous function; specifically it's pointwise limit is zero on $[0,1)$ and 1 at 1. hence we see □

4.0.19 Problem 3

Statement: Problem Number 3

Let $C([0,1])$ be the space of continuous function son the nit interval with the uniform norm and let $\mathbb{R}[x]$ be the subspace of polynomials. Give an example of an unbounded linear transformation $T\colon \mathbb{R}[x] \to \mathbb{R}$.

Proof:
Let $T\colon \mathbb{R}[x] \to \mathbb{R}$ be defined by evaluation at r, for some $r > 1$. That is, for some $p \in \mathbb{R}[x], T(p) = p(r)$. It is clear the evaluation mapping is linear, and we can bound its norm below using the polynomial x^n. Notice x^n has norm 1 in the uniform norm, and so $\|T\| \geq \|Tx\| = r^n$. Taking n large shows T has unbounded norm. □

4.0.20 Problem 4

Statement: Problem Number 4

Let X be a metric space. Prove or disprove the following:

(a) If X is compact, then X is complete.

(b) If X is complete, then X is compact.

Proof:
The first statement is true; the second false. To see compactness implies complete, we use the sequential characterization of compactness. Now let x_n be a Cauchy sequence in X. Then by compactness, x_n contains a subsequence $x_{k(n)}$ which converges to some x_0 in X. Hence $x_n \to x_0$, for

$$|x_n - x_0| \leq |x_n - x_{k(n)}| + |x_{k(n)} - x_0|.$$

Therefore x_n has a limit in X, and so X is complete. Now complete does not imply compact in a metric space. A familiar counterexample is the real numbers, which are certainly not compact, and are complete (though we do not prove this). □

4.0.21 Problem 5

Statement: Problem Number 5

Let H be a Hilbert space and V a linear subspace. Show that $V^{\perp\perp} = \overline{V}$.

Proof:
Recall that $V^{\perp}$ is a closed linear space; this follows immediately from the linearity and continuity of $\langle \cdot, \cdot \rangle$. Next recall there is a direct sum orthogonal decomposition of the Hilbert space $H = V^{\perp} \oplus V^{\perp\perp}$. Our final observation is that $V^{\perp} = (\overline{V})^{\perp}$; again this follows directly from properties of the inner product. Thus we have $H = \overline{V} \oplus (\overline{V})^{\perp} = \overline{V} \oplus V^{\perp}$. Then the isomorphism $V^{\perp} \oplus \overline{V} \cong V^{\perp} \oplus V^{\perp\perp}$ gives the equality $V = V^{\perp\perp}$. $\square$

4.0.22 Problem 6

Statement: Problem Number 6

State Jensen's inequality. Show that the function $\varphi\colon x \to \log(1/x)$ is convex. Suppose that $a_i \geq 0$ and $p_i > 0$ such that the p_i sum to 1. Prove that

$$a_1^{p_1} \cdots a_n^{p_n} \leq a_1 p_1 + \cdots + a_n p_n.$$

Proof:

Jensen's inequality is as follows. Let (X, μ) be a finite measure space, with $\Omega = \int_X 1\mu < \infty$. Let $f\colon X \to \mathbb{R}$ be integrable, and $\varphi\colon \mathbb{R} \to \mathbb{R}$ a convex function. Then

$$\varphi(\frac{1}{\Omega}\int_X f d\mu) \leq \frac{1}{\Omega}\int_X \varphi \circ f d\mu$$

To see that $\varphi\colon (0,\infty) \to \mathbb{R}$ as defined above is convex, notice it is twice differentiable. Then as $\varphi'' = 1/x^2 > 0$, it is a convex function.

Finally, consider the measure space $X = \{p_i\}$ with measure $\int_X f d\mu = \sum f(p_i)p_i$. Let φ be as defined above, and notice the function $f\colon X \to \mathbb{R}$ defined by $f(p_i) = a_i$ is integrable. From Jensen's inequality we get

$$-\log(\sum_i a_i p_i) \leq \sum_i -p_i \log a_i.$$

Exponentiating both sides gives the desired inequality. □

Chapter 5

Winter 2005

Author: Jeffrey Anderson

5.0.23 Problem 1

Statement: Problem 1

Show that the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \frac{\pi}{2} + x - \arctan(x)$$

has no fixed points in $\mathbb{R}$ and that

$$|T(x) - T(y)| \leq |x - y| \qquad \text{for all distinct } x, y \in \mathbb{R}$$

Why does this example not contradict the contraction mapping theorem?

Proof:
Suppose hoping for contradiction that there exists a fixed point for the map $T : \mathbb{R} \to \mathbb{R}$. Then, by definition of fixed point, there is some $x \in \mathbb{R}$ such that

$$\begin{aligned} T(x) &= x \\ &\Rightarrow T(x) = \frac{\pi}{2} + x - \arctan(x) = x \\ &\Rightarrow \frac{\pi}{2} - \arctan(x) = x - x \\ &\Rightarrow \arctan(x) = \frac{\pi}{2} \end{aligned}$$

Then, under our assumption, we have that their exists some $x \in \mathbb{R}$ such that

$$\tan(\frac{\pi}{2}) = x$$

This is not possible, since tan is not defined at $\frac{\pi}{2}$.

Now, let us consider

$$|T(x) - T(y)|$$

for any two distinct $x, y \in \mathbb{R}$. Without loss of generality assume that $x < y$. We know by the mean value theorem, we have there exists some $z \in (x, y)$ such that

$$\|T(x) - T(y)\| = T'(z)|x - y|$$

Then, we also have that for any $x \in \mathbb{R}$

$$\begin{aligned} T'(x) &= \frac{d}{dx}\left(\frac{\pi}{2} + x - \arctan(x)\right) \\ &= 0 + 1 - \frac{1}{1 + x^2} \end{aligned}$$

We can then bound $T'(z) < 1$ since $\frac{1}{1+z^2} \geq 0$. We conclude that

$$\begin{aligned} |T(x) - T(y)| &= T'(z)|x - y| \\ &< 1|x - y| \\ &= |x - y| \end{aligned}$$

This proves the second part of our claim.

To substantiate the last claim of this problem, consider the statement of the Contraction mapping theorem. In the mapping defined above, there is no $c \in (0, 1)$ such that

$$|T(x) - T(y)| < c|x - y|$$

for all $x, y \in \mathbb{R}$.

Assume, hoping for contradiction, that the exists a $c \in (0, 1)$ such that

$$|T(x) - T(y)| < c|x - y|$$

(ie assume that the antecedents of the contraction mapping theorem hold). We know by the Mean Value Theorem taught in Math 21A here at UC Davis, for all $x, y \in \mathbb{R}$, there exists some $\xi \in \mathbb{R}$ such that

$$\begin{aligned} |T(x) - T(y)| &= T'(\xi)|x - y| \\ &= \underbrace{1 - \frac{1}{1 + \xi^2}}_{T'(\xi)} |x - y| \end{aligned}$$

We will explicitly exhibit a pair (x, y) which defies this inequality, thus leading to a contradiction and proving that no such c can exist. Let $y = 0$. Then we have

$$\begin{aligned} |T(x) - T(0)| &= \left|T(x) - \frac{\pi}{2}\right| \\ &= \left|\frac{\pi}{2} + x - \arctan(x) - \frac{\pi}{2}\right| \\ &= |x - \arctan(x)| \end{aligned}$$

We note that for $x > 0$ we have $x - \arctan(x) > x - \frac{\pi}{2}$ by the properties of $\arctan(x)$. Now choose

$$x = \frac{\pi}{2(1-c)}$$

We note that $\frac{\pi}{2(1-c)} > 0\pi/2$. Then we have

$$\begin{aligned} |T(x) - T(0)| &= |x - \arctan(x)| \\ &> x - \frac{\pi}{2} \\ &= \frac{\pi}{2(1-c)} - \frac{\pi}{2} \\ &= \frac{\pi c)}{2(1-c)} \\ &= cx \end{aligned}$$

Then, given we have a c, we have found a pair (x, y) of points that defies the bound. This contradicts the assumption that our function T above satisfies the contraction mapping theorem. With this, we have shown each of the three parts of this problem. $\square$

5.0.24 Problem 2

Statement: Problem 2

> Prove that the vector space $C([a,b])$ is seperable. Here and below, $C([a,b])$ is the vector space of continuous functions $f : [a,b] \to \mathbb{R}$ with the supremum norm.

Proof:

This is exactly what we wanted to show. □

5.0.25 Problem 3

Statement: Problem 3

Suppose that $f_n \in C([a,b])$ is a sequence of functions converging uniformly to a function f. Show that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Give a counterexample to show that the pointwise convergence of continuous functions f_n to a continuous function f does not imply convergence of the corresponding integrals.

Proof:
Let $f_n \in C([a,b])$ be a sequence of functions converging uniformly to a function f. Let $\epsilon > 0$. Since f_n converges uniformly, we have that there is an $N \in \mathbb{N}$ such that for all $n \geq N$

$$|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$$

for all $x \in [a,b]$. Assuming $n \geq N$, we have

$$\begin{aligned}
\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| &= \left|\int_a^b (f_n(x) - f(x))dx\right| && \text{by linearity of the integral} \\
&\leq \int_a^b |f_n(x) - f(x)|dx && \text{bringing absolute value inside integral} \\
&< \int_a^b \epsilon(b-a)dx && \text{by assumption on } n \\
&= \frac{\epsilon}{b-a}\int_a^b 1dx \\
&= \frac{\epsilon}{b-a}(b-a) \\
&= \epsilon
\end{aligned}$$

This proves the first part of our problem. For the second part of this problem, consider the sequence $f_n \in C([0,1])$ which linearly interpolates the following secquence of points

- for $x \in (0, 1/2^n)$, f_n is defined by the line through the points $(0,0)$ and $(1/2^n, 2^n)$
- for $x \in (1/2^n, 1/2^{n-1})$, f_n is defined by the line through the points $(1/2^n, 2^n)$ and $(1/2^{n-1}, 0)$

- for $x \in (1/2^{n-1}, 0)$, f_n is defined by the constant function 0.

To explicitly formulate this sequence of functions, we rely on the following: Given two points (x_1, y_1) and (x_2, y_2) we can define the interpolating line by computing

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$(y - y_1) = m(x - x_1)$$

Then, this sequence of function is defined by

$$f_n(x) = \begin{cases} 2^{2n}x & \text{if } 0 \leq x < 2^{-n} \\ -2^{2n}(x - 1/2^n) + 2^n & \text{if } 2^{-n} \leq x < 2^{-(n-1)} \\ 0 & \text{if } 2^{-(n-1)} \leq x \leq 1 \end{cases}$$

We notice that this sequence of functions converges pointwise to zero, yet the integral of each f_n is one. This demonstrates exactly what we would like. $\square$

5.0.26 Problem 4

Statement: Problem 4

Let $l_2(\mathbb{Z})$ denote the complex Hilbert space of sequences $x_n \in C$, $n \in \mathbb{Z}$ such that

$$\sum_{n=-\infty}^{\infty} |x_n|^2 < \infty$$

Define the *shift operator* $S : l_2(\mathbb{Z}) \to l_2(\mathbb{Z})$ by

$$S((x_n)) = (x_{n+1})$$

Show that S has no eigenvalues.

Proof:
Let $x_n \in \ell_2(\mathbb{Z})$ as described above. Assume, hoping for contradiction, that the shift operator does have an eigenvalue $\lambda \in \mathbb{C}$. By definition we have the following equalities:

$$\begin{aligned} S((x_0)) &= x_{0+1} = x_1 = \lambda x_0 \\ S((x_1)) &= x_{1+1} = x_2 = \lambda x_1 = \lambda^2 x_0 \\ S((x_2)) &= x_{2+1} = x_3 = \lambda x_2 = \lambda^3 x_0 \end{aligned}$$

With this initial observation, we see that the assumption that $\lambda \in \mathbb{C}$ is an eigenvalue immediately mandates a very unique structure to the sequence $\{x_n\}_{n=-\infty}^{\infty}$. Specifically, assume that n is a positive integer. Then

$$x_n = \lambda^n x_0$$

Similarly if m is a negative integer, then

$$x_m = \lambda^m x_0$$

Without loss of generality, we can assume $|x_0| = 1$ (if not, we know that the sequence convergence and hence we can use the constant multiple rule for converging sequences taught in Math 21C here at UC Davis and normalize to enforce this condition).

Then we have

$$\sum_{-\infty}^{\infty} |x_n|^2 = \sum_{-\infty}^{\infty} |\lambda|^{2n}$$

The sequence $\{x_n\}_{n=-\infty}^{\infty}$ illustrates that $\lambda = 0$ is can never be an eigenvalue. Similarly, if $0 < |\lambda| < 1$ we have that the sum

$$\sum_{-\infty}^{0} |\lambda|^{2n}$$

diverges. Similarly, if $|\lambda| > 1$, we know that

$$\sum_{0}^{\infty} |\lambda|^{2n}$$

diverges because it fails the test for convergence. Last, if $\lambda = 1$, we have that the sum of all ones diverges because it fails the convergence test. We have reached a contradiction for any $\lambda \in \mathbb{C}$. It must be that S has no eigenvalues.s $\square$

5.0.27 Problem 5

Statement: Problem 5

Consider the initial value problem

$$u'(t) = |u(t)|^\alpha, \qquad u(0) = 0$$

Show that the solution to this problem is unique if $\alpha > 1$ and not unique if $0 \leq \alpha < 1$

Proof (exercise 2.13 in Applied Analysis):
Suppose that $\alpha > 1$. In this case we reference theorem 2.26 of Applied Analysis. We must check that the antecedents of this theorem are preserved:

In this case, let us define

$$f(t, u) = u'(t) = |u(t)|^\alpha$$

We note that this function is continuous on the rectangle

$$R = \{|t| < T, |u| < L\}$$

With this, we see that f is Lipschitz and we get that the solution of our IVP is unique.

Let $0 \leq \alpha < 1$. We will validate the second claim by direct computation. Specifically, we find that

$$u(t) = \begin{cases} 0 & t \leq a \\ (a-\alpha)^{\frac{1}{1-\alpha}}(t-\alpha)^{\frac{1}{1-\alpha}} & t \geq a \end{cases}$$

where a is some constant. We also note that $u(t) = 0$ is a solution to the same initial value problem. We have explicitly listed two seperate solutions and thus have shown our second desired property.

Suppose that $\alpha > 1$. In this case We see that This is exactly what we wanted to show. □

5.0.28 Problem 6

Statement: Problem 6

Let $\mathcal{H}$ be a Hilbert Space, $\mathcal{H}_0$ a dense linear subspace of $\mathcal{H}$, $\{x_n\}_{n=1}^{\infty} \subset H$ and $x \in \mathcal{H}$ such that

i. there exists $M > 0$ such that $\|x_n\| \leq M$ for all n

ii. $\lim_{n\to\infty} \langle x_n, y\rangle = \langle x, y\rangle$ for all $y \in \mathcal{H}_0$

Prove that $\{x_n\}_{n=1}^{\infty}$ converges in the weak topology of $\mathcal{H}$

Proof: Theorem 8.40 in Applied Analysis
Let all the assumptions in the problem statement hold. Let $y \in \mathcal{H}$. Let $\epsilon > 0$. We want to show that for that

$$\lim_{n\to\infty} \langle x_n, y\rangle = \langle x, y\rangle$$

We can rewrite this in the form

$$\lim_{n\to\infty} \langle x_n - x, y\rangle = 0$$

using the definition of the inner product. We will prove our statement in this form.

Since $y \in \mathcal{H}$ and $\mathcal{H}_0$ is dense in $\mathcal{H}$, we have that there is some $\chi \in \mathcal{H}_0$ such that $\|\chi - y\| < \epsilon$ (this is an immediate consequence of the definition of density). Since $\chi \in \mathcal{H}_0$, we know by our assumptions above

$$\lim_{n\to\infty} \langle x_n, \chi\rangle = \langle x, \chi\rangle$$

We can rearrange this using the same ideas as above to see

$$\lim_{n\to\infty} \langle x_n - x, \chi = 0 \tag{5.1}$$

We see that there is some postive integer N such that

$$|\langle x_n - x, \chi\rangle| < \epsilon$$

for all $n > N$ by the definition of weak convergence.

Now let us consider

$$\begin{aligned}
|\langle x_n - x, y\rangle| &= |\langle x_n - x, y - \chi + \chi\rangle| \\
&= |\langle x_n - x, y - \chi\rangle| + |\langle x_n - x, \chi\rangle| \\
&\leq |\langle x_n - x, y - \chi\rangle| + \epsilon && \text{using the assumption (5.1)} \\
&\leq \|x_n - x\|\|\chi - y\| + \epsilon && \text{applying Cauchy Schwarz Inequality} \\
&\leq (\|x_n\| + \|x\|)\|\chi - y\| + \epsilon && \text{by the triangle inequaltiy} \\
&\leq (M + \|x\|)\|\chi - y\| + \epsilon && \text{since } x_n \text{ is bounded} \\
&\leq (M + \|x\|)\epsilon + \epsilon && \text{by choice of } \chi
\end{aligned}$$

This is exactly what we wanted to show. □

Chapter 6

Fall 2007

6.0.29 Problem 1

Author: David Renfrew

Statement: Problem 1

Suppose that $f : [0,1] \to \mathbb{R}$ is continuous. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x^n)dx$$

exists and evaluate the limit. Does the limit alway exist if f is only assumed to be integrable?

Proof:
Let $g(x) := \sup_x f(x)$, because f is continuous this exist and g is integrable with $f(x) \leq g(x)$ so we can apply the LDCT:

$$\lim_{n\to\infty} \int_0^1 f(x^n)dx = \int_0^1 \lim_{n\to\infty} f(x^n)dx = \int_0^1 f(\lim_{n\to\infty} x^n)dx = f(0)$$

The limit does not necessiarly exist if f is only integrable.
Consider $f(x) = x^{-1/2}$ which is integrable but

$$\int_0^1 f(x^n)dx = \int_0^1 x^{-n/2}dx = \left(\frac{1}{1-n/2}\right) x^{1-n/2}\Big|_0^1$$

This integral does not converge for $n > 1$.

This establishes our claim □

6.0.30 Problem 2

Author: David Renfrew

6.0.31 Problem 1

Author: David Renfrew

Statement: Problem 2

Suppose that for each $n \in \mathbb{Z}$, we are given a real number ω_n. For each $t \in \mathbb{R}$, define a linear operator $T(t)$ on 2π-periodic function by

$$T(t)\left(\sum_{n\in\mathbb{Z}} f_n e^{inx}\right) = \sum_{n\in\mathbb{Z}} e^{i\omega_n t} f_n e^{inx},$$

where $f(x) = \sum_{n\in\mathbb{Z}} f_n e^{inx}$ with $f_n \in \mathbb{C}$.
(a) Show that $T(t) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a unitary map.
(b) Show that $T(s)T(t) = T(s+t)$ for all $s, t \in \mathbb{R}$.
(c) Prove that if $f \in C^\infty(\mathbb{T})$, meaning that it has continuous derivative of all orders, then $T(t)f \in C^\infty(\mathbb{T})$.

Proof:
(a) First observe by the density of the trig polynomials $T(t)$ is onto.
Then let $f, g \in L^2(\mathbb{T})$.
Then compute the inner product using the Fourier coefficients.

$$\begin{aligned}
\langle T(t)f, T(t)g\rangle &= \sum_{n\in\mathbb{Z}} \left(\overline{e^{i\omega_n t} f_n}\right)\left(e^{i\omega_n t} g_n\right) \\
&= \sum_{n\in\mathbb{Z}} \overline{f_n} g_n \\
&= \langle f, g\rangle
\end{aligned}$$

(b) By direct computation:

$$\begin{aligned}
T(s)T(t)f &= T(s)\sum_{n\in\mathbb{Z}} e^{i\omega_n t} f_n e^{inx} \\
&= \sum_{n\in\mathbb{Z}} e^{i\omega_n s} e^{i\omega_n t} f_n e^{inx} \\
&= \sum_{n\in\mathbb{Z}} e^{i\omega_n s+t} f_n e^{inx} \\
&= T(s+t)f
\end{aligned}$$

(c) This follows by Sobolev embedding.

6.0.32 Problem 3

Author: David Renfrew

Statement: Problem 3

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be Banach spaces, with X compactly imbedded in Y, and Y continuously imbedded in Z (meaning: $X \subset Y \subset Z$; bounded sets in $(X, \|\cdot\|_X)$ are precompact in $(Y, \|\cdot\|_Y)$; and there is a constant M such that $\|x\|_Z \leq M\|x\|_Y$ for every $x \in Y$). Prove that for every $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that

$$\|x\|_Y \leq \epsilon\|x\|_X + C(\epsilon)\|x\|_Z \quad \text{for every } x \in X.$$

Proof:

Assume for contradiction that there is an ϵ for which such a $C(\epsilon)$ does not exist. Then there exist a sequence x_n with X-norm 1 such that

$$\|x_n\|_Y > \epsilon + n\|x_n\|_Z$$

By the compact embedding the x_n's have a convergent subsequence in the Y norm, let us pass to this subsequence.

This means $\|x_n\|_Y$ will converge to a constant, but this means the above inequality will only be preserved if $\|x_n\|_Z$ goes to zero.

This implies $x_n \to 0$ in all spaces. This contradicts the assumption that $\|x_n\|_Y > \epsilon$, so the desired inequality is proved.

6.0.33 Problem 4

Author: David Renfrew

Statement: Problem 4

Let $\mathcal{H}$ be the weighted L^2-space

$$\mathcal{H} = \left\{ f : \mathbb{R} \to \mathbb{C} \middle| \int_R e^{-|x|} |f(x)|^2 dx < \infty \right\}$$

with inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} e^{-|x|} \overline{f(x)} g(x) dx.$$

Let $T : \mathcal{H} \to \mathcal{H}$ be the translation operator

$$(Tf)(x) = f(x+1).$$

Compute the adjoint T^* and the operator norm $\|T\|$.

Proof:
First the adjoint computation: Let $f, g \in \mathcal{H}$.

$$\begin{aligned}
\langle Tf, g \rangle &= \int_{\mathbb{R}} e^{-|x|} \overline{Tf(x)} g(x) dx \\
&= \int_{\mathbb{R}} e^{-|x|} \overline{f(x+1)} g(x) dx \\
&= \int_{\mathbb{R}} e^{-|x-1|} \overline{f(x)} g(x-1) dx \\
&= \int_{\mathbb{R}} e^{-|x|} \overline{f(x)} e^{|x|-|x-1|} g(x-1) dx \\
&= \langle f, e^{|x|-|x-1|} g(x-1) \rangle
\end{aligned}$$

So $T * g(x) = e^{|x|-|x-1|} g(x-1)$.

Now we compute the norm:

6.0.34 Problem 5

Author: David Renfrew

Statement: Problem 5

a) State the Rellich Compactness Theorem for the space $W^{1,p}(\Omega)$ for $\Omega \subset \mathbb{R}^n$.
(b) Suppose that $\{f_n\}_{n=1}^{\infty}$ is a bounded seqeunce in $H^1(\Omega)$ for $\Omega \subset \mathbb{R}^3$ open, bounded, and smooth. Show that there exists an $f \in H^1(\Omega)$ such that for a subsequence $\{f_{n_l}\}_{l=1}^{\infty}$,

$$f_{n_l} Df_{n_l} \rightharpoonup fDf \quad \text{weakly in } L^2(\Omega),$$

where $D = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ denotes the weak gradient operator.

Proof:

(a) Let $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain. Let $1 \leq p \leq n$ and $p^* := \frac{np}{n-p}$.
Then $W^{1,p}(\Omega)$ is compacted embedded in $L^q(\Omega)$ for all $1 \leq q < p^*$.

(b)
Since f_n is bounded in H^1 so f_n and Df_n are bounded in L^2.
Bounded sets are weakly precompact so after passing to a subsequence there exist an f and g such that $f_n \rightharpoonup f$ and $Df_n \rightharpoonup g$. Additionally $g = Df$.

$$f_n Df_n = (f_n - f)Df_n + fDf_n$$

The Sobolev conjugate $p* = \frac{np}{n-p} = \frac{3*2}{3-2} = 6 > 2$, so we can apply Rellich's Theorem, after passing to a subseqence, to say that $f_{n_l} \to f$ strongly in L^2.
Since Df_n are bounded $(f_n - f)Df_n$ converges to 0 in L^2. So we have:

$$\begin{aligned}\lim_{l\to\infty} f_{n_l} Df_{n_l} &= \lim_{l\to\infty} (f_{n_l} - f)Df_{n_l} + fDf_{n_l} \\ &\rightharpoonup 0 + fDf\end{aligned}$$

As desired.

Author: David Renfrew

Statement: Problem 6

Let $\Omega := B(0, \frac{1}{2}) \subset \mathbb{R}^2$ denote the open ball of radius $\frac{1}{2}$. For $x = (x_1, x_2) \in \Omega$, let

$$u(x_1, x_2) = x_1 x_2 \left[\log\left(|\log(|x|)|\right) - \log\log 2\right] \quad \text{where } |x| = \sqrt{x_1^2 + x_2^2}.$$

(a) Show that $u \in C^1(\overline{\Omega})$.
(b) Show that $\frac{\partial^2 u}{\partial x_j^2} \in C(\overline{\Omega})$. for $j = 1, 2$, but that $u \notin C^2(\overline{\Omega})$.
(c) Using the elliptic regularity theorem for the Dirichlet problem on the disc, show that $u \in H^2(\Omega)$.

Proof:

First we compute the derivaties:

$$\frac{\partial}{\partial x_1} u(x_1, x_2) = x_2 \left[\log\left(|\log(|x|)|\right) - \log\log 2\right] + \frac{x_1 x_2}{|\log(|x|)|} \frac{\log(|x|)}{|\log(|x|)|} \frac{1}{|x|} \frac{x_1}{|x|}$$

$$\frac{\partial}{\partial x_2} u(x_1, x_2) = x_1 \left[\log\left(|\log(|x|)|\right) - \log\log 2\right] + \frac{x_1 x_2}{|\log(|x|)|} \frac{\log(|x|)}{|\log(|x|)|} \frac{1}{|x|} \frac{x_2}{|x|}$$

Chapter 7

Winter 2008

7.0.35 Problem 1

Author: Jeffrey Anderson

Statement: Problem Number

Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = (-1)^n x^n (1-x)$$

a. Show that $\sum_{n=0}^{\infty} f_n$ converges uniformly on $[0,1]$

b. Show that $\sum_{n=0}^{\infty} |f_n|$ converges pointwise on $[0,1]$ but not uniformly

Proof:

7.0.36 Problem 2

Author: Jeffrey Anderson

7.0.37 Problem 3

Author: Jeffrey Anderson

Statement: Problem 3

> Suppose that $\mathcal{M}$ is a (nonzero) closed linear subspace of a Hilbert space $\mathcal{H}$ and $\phi : \mathcal{M} \to \mathbb{C}$ is a bounded linear functional on $\mathcal{M}$. Prove that there is a unique extension of ϕ to a bounded linear function on $\mathcal{H}$ with the same norm.

Proof:
The first thing to notice is that this problem is known as the Hahn-Banach Theorem for Closed linear subspaces. With this in mind, the way to establish this result is:

- construct the norm preserving extension explicitly to the larger subspace
- Invoke the axiom of choice to establish that the construction above gives our desired extension

Then, let $\mathcal{H}$ be a Hilbert space, let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$, let $\phi : \mathcal{M} \to \mathbb{C}$ be a bounded linear functional on $\mathcal{M}$. We want to show

i. there exists a norm preserving extension of ϕ

ii. this extension is unique

First we will show that an extension exists. Let $x_0 \in \mathcal{H}$ such that $x_0 \notin \mathcal{M}$, Let

$$Y_1 = \langle x_0, \mathcal{M} \rangle = \text{span}\{x_0, \mathcal{M}\}$$

We know that for any $y \in Y_1$, we have $y = x + \lambda x_0$ for $x \in M, \lambda \in \mathbb{C}$. Let us define a mapping $\Phi : Y_1 \to \mathbb{C}$ by

$$\begin{aligned} \Phi(y) &= \Phi(x + \lambda x_0) \\ &= \phi(x) + \lambda\alpha \end{aligned}$$

for some $\alpha \in \mathbb{C}$.

We can check that this mapping is linear:

$$\begin{aligned} \Phi(y + z) &= \Phi\left((x + \lambda x_0) + (w + \mu w_0)\right) \\ &= \phi(x + w) + (\lambda + \mu)\alpha \\ &= \phi(x) + \lambda\alpha + \phi(w) + \mu\alpha \\ &= \Phi(y) + \Phi(z) \end{aligned}$$

The trick of this problem is to choose α so that the extended funcitonal has norm 1.

Recall by definition

$$\begin{aligned} \|\Phi\|_{op} &= \sup_{\|y\|=1} \Phi(y)\|_H \\ &= \sup_{\|x+\lambda x_0\|=1} \|\phi(x) + \lambda\alpha\|_{\mathbb{C}} \end{aligned}$$

We can bound $\|\Phi\|_{op}$ above by one using the inequalities

$$\begin{aligned}\|\Phi(y)\|_{op} &= \|\phi(x) + \lambda\alpha\|_{\mathbb{C}} \\ &\leq \|x + \lambda x_0\|_H \\ &= \|y\|_H\end{aligned}$$

Now, to find the lower bound, take $x = -\lambda x \in \mathcal{M}$. Then

$$\begin{aligned}|\phi(-\lambda x) + \lambda\alpha| &= |-\lambda\phi(x) + \lambda\alpha| \\ &\leq \|-\lambda x + \lambda x_0\|_H\end{aligned}$$

This is true if and only if

$$\begin{aligned}&|\lambda||\phi(x) - \alpha| \leq |\lambda|\|x - x_0\|_H \\ \Leftrightarrow\ &|\phi(x) - \alpha| \leq \|x - x_0\|_H\end{aligned}$$

From here, we would like to solve for α such that

$$\begin{aligned}&-\|x - x_0\|_H \leq \phi(x) - \alpha \leq \|x - x_0\|_H \\ \Leftrightarrow &-\phi(x) - \|x - x_0\|_H \leq -\alpha \leq -\phi(x) + \|x - x_0\|_H \\ \Leftrightarrow &\alpha \in [\Phi(x) - \|x - x_0\|, \Phi(x) - \|x - x_0\|]\end{aligned}$$

We introduce the notation

$$\begin{aligned}Ax &= \Phi(x) - \|x - x_0\| \\ Bx &= \Phi(x) + \|x - x_0\|\end{aligned}$$

We also let $\Sigma = [Ax, Bx] \subset \mathbb{R}$. We note that $\Sigma \neq \emptyset \iff Ax \leq Bx$ for all $x, y \in \mathcal{M}$. We note that

$$\begin{aligned}\phi(x) - \phi(y) &= \phi(x - y) \\ &= \|x - y\| \\ &= \|x + x_0 - x_0 - y\| \\ &\leq \|x - x_0\| + \|y - x_0\|\end{aligned}$$

Then we have

$$\phi(x) - \|x - x_0\| \leq \phi(y) + \|y - x_0\|$$

With this we see there is an extension that perserves the norm. We must now establish uniqueness.

7.0.38 Problem 4

Author: Mohammad Omar

7.0.39 Problem 5

Author: Jeffrey Anderson

Statement: Problem 5

Let $1 \leq p < \infty$ and let $I = (-1, 1)$ denote the open interval in $\mathbb{R}$. Find the values of α as a function of p for which the function $|x|^\alpha \in W^{1,p}(I)$

Proof:

We want to find the values of α as a function of p such that $(|x|)^\alpha \in W^{1,p}(I)$. By the definition of the space $W^{1,p}(I)$, we know we must solve three related problems. These problems include

(a) Find the values of α in terms of p such that $(|x|)^\alpha \in L^p(I)$

(b) Find the values of α in terms of p such that $\frac{d}{dx}(|x|)^\alpha \in L^p(I)$ (where we are taking the weak derivative here)

(c) Compare these values of α to find the correct interval we desire

First, by the definition of the L^p-norm, we have

$$\|(|x|)^\alpha\|_p = \left(\int_{-1}^{1} |(|x|)^\alpha|^p dm \right)^{1/p}$$
$$= \left(\int_{0}^{1} x^{p\alpha} dm \right)^{1/p} \qquad \text{by symmetry of } |x| \text{ around } x = 0$$

We know by a theorem discussed in Math 21B here at UC Davis

$$\int_1^\infty x^\beta dm < \infty \Leftrightarrow \beta \in (-1, \infty)$$

From here we notice that there is a relationship between the integrability of $x^{1/\beta}$ on $(0, 1)$ and the integrability of x^β on the interval $(1, \infty)$.

Then

$$\int_0^1 x^\mu dm < \infty \Leftrightarrow \mu \in (-1, 0) \cup [0, \infty)$$

We conclude that $x^{p\alpha}$ has finite integral on $(0, 1)$ if and only if $p\alpha > -1$ or $\alpha > \frac{-1}{p}$

Second, we have that if the weak derivative of this function exists, it is equal (almost everywhere) to the classical derivative of our function. We know, by definition

$$(|x|)^\alpha = \begin{cases} (-x)^\alpha & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x^\alpha & \text{if } x > 0 \end{cases}$$

This function is classically differentiable everywhere except $x = 0$. The classic derivative of this function using methods taught in Math 21A here at UC Davis is given by

$$\frac{d}{dx}(|x|)^\alpha = \begin{cases} -\alpha(-x)^{\alpha-1} & \text{if } x < 0 \\ \alpha x^{\alpha-1} & \text{if } x > 0 \end{cases}$$

Now we take the L^p norm of the weak derivative $\partial(|x|)^\alpha$:

$$\begin{aligned} \|\partial(|x|)^\alpha\|_p &= \left(\int_{-1}^{1} |\alpha(|x|)^{\alpha-1}|^p dm \right)^{1/p} \\ &= 2\alpha \left(\int_{0}^{1} (x)^{p(\alpha-1)} dm \right)^{1/p} \end{aligned}$$

We conclude

$$\begin{aligned} \|\partial(|x|)^\alpha\|_p < \infty &\Leftrightarrow p(\alpha - 1) > -1 \\ &\Leftrightarrow \alpha > 1 - \frac{1}{p} \end{aligned}$$

Last, we conclude that the only way that $|x|^\alpha \in W^{1,p}(I)$ is if $\alpha > 1 - \frac{1}{p}$

7.0.40 Problem 6

Author: Mohammad Omar

Chapter 8

Fall 2008

8.0.41 Problem 1

Author: Bailey Meeker

Statement: Problem 1

Prove that the dual space of c_0 is ℓ^1, where $c_0 = \left\{ \{x_n\}_{n=1}^{\infty} | \lim_{n\to\infty} x_n = 0 \right\}$

Proof:
We can restate this problem as such: Let $T \in c_0^*$. Then there is a $\{a_n\}_{n=1}^{\infty} \in \ell^1$ such that $T(\{x_n\}_{n=1}^{\infty}) = \sum x_n a_n$ for all $\{x_n\}_{n=1}^{\infty} \in \ell^1$.

Let $\delta_{m,n}(x) = \begin{cases} 1 & : n = m \\ 0 & : n \neq m \end{cases}$. Let $a_n = T(\delta_n)$ where $\delta_n = (0, 0, \ldots, 0, 1, 0, \ldots)$ with the 1 appearing in the nth coordinate. Let $\{x_n\}_{n=1}^{\infty} \in c_0$. Finally, define $\{y_n\}_{n=1}^{\infty}$ by the folowing: $y_n = \sum_{k=1}^{n} x_k \delta_k$.

Step 1: Show that T can be identified with some sequence a_n.

By the definition above,

$$T(y_n) = T\left(\sum_{k=1}^{n} x_k \delta_k\right) = \sum_{k=1}^{n} T(x_k \delta_k) = \sum_{k=1}^{n} x_k T(\delta_k) = \sum_{k=1}^{n} x_k a_k.$$

Note that $\sum_{k=1}^{n} T(x_k \delta_k) = \sum_{k=1}^{n} x_k T(\delta_k)$ since T is a bounded linear functional on c_0, and thus scalar multiples come through. Consider

$$y_n - \{x_n\}_{n=1}^{\infty} = \sum_{k=1}^{n} x_k \delta_k - \{x_n\}_{n=1}^{\infty} = (x_1, x_2, \ldots, x_n, 0, \ldots) - \{x_n\}_{n=1}^{\infty}$$

$$= (x_1, x_2, \ldots, x_n, 0, \ldots) - \{x_n\}_{n=1}^{\infty} = (0, \ldots, 0, -x_{n+1}, -x_{n+2}, \ldots).$$

Then consider the c_0 norm of this difference:

$$\|y_n - \{x_n\}_{n=1}^{\infty}\|_{c_0} = lub\,|y_n - x_{n}{}_{n=1}^{\infty}| = lub\{|x_k|\,|k \geq n\}.$$

Then if we take the limit as $n \to \infty$ we see that

$$lim_{n\to\infty} \|y_n - \{x_n\}_{n=1}^{\infty}\|_{c_0} = lim_{n\to\infty} lub\{|x_k|\,|k \geq n\} = limsup_{n\to\infty}\{x_n\}_{n=1}^{\infty} = lim_{n\to\infty} x_n = 0.$$

Therefore $lim_{n\to\infty} y_n = \{x_n\}_{n=1}^{\infty}$ in c_0. We assumed that T is a bounded linear functional, therefore T is continuous. Then

$$T(\{x_n\}_{n=1}^{\infty}) = T(lim_{n\to\infty} y_n) = lim_{n\to\infty} T(y_n) = lim_{n\to\infty} \sum_{k=1}^{n} x_k a_k = \sum_{k=1}^{\infty} x_k a_k.$$

Step 2: Show that the sequence $\{a_n\}_{n=1}^{\infty} \in \ell^1$.

To prove this, we will use the flowing sequence: $\{\gamma_{m,n}\}_{m=1}^{\infty}$ where

$$\gamma_{m,n} = \begin{cases} 1 & m \leq n \text{ and } a_m \geq 0 \\ 0 & m \leq n \text{ and } a_m < 0 \\ 0 & m > n \end{cases}$$

Consider

$$T(\{\gamma_{m,n}\}_{m=1}^{\infty}) = \sum_{k=1}^{n} \gamma_{k,n} a_k = \sum_{k=1}^{n} |a_k|$$

by construction. We know that $\{\gamma_{m,n}\}_{m=1}^{\infty} \in c_0$ since $\gamma_{m,n} = 0$ for all $n > m$. Specifically we note that $\|\gamma_{m,n}\|_{c_0} = 1$ by construction.

□

8.0.42 Problem 2

Author: Mohamed Omar

Statement: Problem 2

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of differentiable functions on a finite interval $[a, b]$ such that the function themselves and their derivatives are uniformly bounded on $[a, b]$. Prove that $\{f_n\}_{n=1}^{\infty}$ has a uniformly converging subsequence.

Proof:

Since $[a, b]$ is a compact subset of $\mathbb{R}$, the Arzela-Ascoli Theorem implies that $\{f_n\} \subset C([a, b])$ has a uniformly convergent subsequence if we can prove it is both uniformly bounded and equicontinuous. We are given that $\{f_n\}$ is uniformly bounded, so it suffices to show it is equicontinuous. We prove the stronger result that the $\{f_n\}$ is uniformly equicontinuous.

Since $\{f_n'\}$ are uniformly bounded, there is a constant $C > 0$ such that $|f_n'(x)| \leq C$ for all n and for all $x \in [a, b]$. Now let $\epsilon > 0$, f_n and $x, y \in [a, b]$ be arbitrary and such that $|x - y| < \frac{\epsilon}{C}$. By the Mean Value Theorem, there exists $c \in (x, y)$ such that $f_n'(c) = \frac{f_n(x) - f_n(y)}{x - y}$. From this we have

$$|f_n(x) - f_n(y)| = |f_n'(c)||x - y| < C\left(\frac{\epsilon}{C}\right) = \epsilon.$$

This argument is independent of our choice of x, y and n, so we conclude $\{f_n\}$ is uniformly equicontinuous. □

8.0.43 Problem 3

Author: Mohamed Omar

Statement: Problem 3

Let $f \in L^1(\mathbb{R})$ and V_f be the closed subspace generatred by the translates of $f : \{f(\cdot - y) | \forall y \in \mathbb{R}\}$. Suppose $\tilde{f}(\xi_0) = 0$ for some ξ_0. Show that $\tilde{h}(\xi_0) = 0$ for all $h \in V_f$. Show that if $V_f = L^1(\mathbb{R})$, then $\tilde{f}$ never vanishes.

Proof:

Recall that, upto scaling,

$$\hat{f}(\xi_0) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi_0} \, dx.$$

Now suppose $\hat{f}(\xi_0) = 0$. Then for any $h = f(\cdot - y)$ for $y \in \mathbb{R}$, we have

$$\begin{aligned}
\hat{h}(\xi_0) &= \int_{\mathbb{R}} f(x-y) e^{-2\pi i x \xi_0} \, dx \\
&= \int_{\mathbb{R}} f(x) e^{-2\pi i (x+y) \xi_0} \, dx \\
&= e^{-2\pi i y \xi_0} \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi_0} \, dx \\
&= e^{-2\pi i y \xi_0} \hat{f}(\xi_0) \\
&= 0.
\end{aligned}$$

Let W_f be the linear space spanned by functions $h = f(\cdot - y)$ for $y \in \mathbb{R}$. Since the Fourier Transform is linear, it follows $\hat{h}(\xi_0) = 0$ for all $h \in W_f$. Now any function $h \in V_f$ is either in W_f (in which case we just showed $\hat{h}(\xi_0) = 0$), or is the L^1 limit of functions $\hat{h}_n \in W_f$, in which case $\hat{h}(\xi) = 0$ because L^1 convergence implies pointwise convergence. □

8.0.44 Problem 4

Author: Mohamed Omar

Statement: Problem 4

a. State the Stone-Weierstrass theorem for a compact Hausdorff space X.

b. Prove that the algevra generated by function so the form

$$f(x, y) = g(x)h(y)$$

where $g, h \in C(X)$ is dense in $C(X \times X)$.

Proof:

[a.] Let X be a compact Hausdorff space, and A be a subalgebra of $C(X)$ which contains a non-zero constant function, and separates points (i.e. for all $x_1, x_2 \in X$, there exists $f \in C(X)$ such that $f(x_1) \neq f(x_2)$). Then A is dense in $C(X)$.

[b.] First note that if X is a compact Hausdorff space, then so is $X \times X$. Let A be the algebra generated by functions of the form $g(x)h(y)$ where $g, h \in C(X)$. Since for any pair $g, h \in C(X)$ we have $g(x)h(y) \in C(X \times X)$, A is a subalgebra of $C(X \times X)$. The constant function $g(x, y) = 1$ for all $(x, y) \in X \times X$ is a member of A since it is the product of the constant function $g(x) = 1$ and itself. By the Stone-Weierstrass Theorem it suffices to show that A separates points. Let $(x_1, y_1), (x_2, y_2) \in X \times X$ be different points. Then at least one of the following is true: $x_1 \neq x_2$ or $y_1 \neq y_2$. Assume without loss of generality that $x_1 \neq x_2$. Since X is Hausdorff, there is an open set U containing x_1 that does not contain x_2. By Urysohn's Lemma, there is a function $g \in C(X)$ that is 1 on U and 0 otherwise. This g, together with the constant function $h = 1$ on X, gives the function $gh \in C(X \times X)$ that separates (x_1, y_1) and (x_2, y_2) (since $gh(x_1, y_1) = 1$ and $gh(x_2, y_2) = 0$). $\square$

8.0.45 Problem 5

Author: Mohamed Omar

Statement: Problem 5

For $r > 0$, define the dilation $d_r f : \mathbb{R} \to \mathbb{R}$ of the function $f : \mathbb{R} \to \mathbb{R}$ by $d_r f(x) = f(rx)$ and the dialation $d_r T$ of the distribution $R \in \mathcal{D}'(\mathbb{R})$ by

$$\langle d_r T, \phi\rangle = \frac{1}{r}\left\langle T, d_{1/r}\phi\right\rangle$$

for all test functions $\phi \in \mathcal{D}(\mathbb{R})$.

- Show that the dialation of a regular distribution T_f, given by

$$\langle T_f, \phi\rangle = \int f(x)\phi(x)dx$$

  agrees with the dilation of the corresponding function f

- A distribution is homogeneous of degree n if $d_r T = r^n T$. Show that the $\delta-$distribution is homogeneous of degree -1

- IF T is a homogeneous distribution of degree n, prove that the derivative T' is a homogeneous distribution of degree $n-1$.

Proof:

[a.] Let $\phi \in D(\mathbb{R})$ be arbitrary. We have

$$\begin{aligned}\langle d_r T_f, \phi\rangle &= \frac{1}{r}\langle T_f, d_{1/r}\phi\rangle \\ &= \frac{1}{r}\int f(x)\phi\left(\frac{x}{r}\right)\ dx \\ &= \int f(rx)\phi(x)\ dx \\ &= \langle T_{rf}, \phi\rangle\end{aligned}$$

Since ϕ is arbitrary, we conclude $d_r T_f = T_{rf}$.

[b.] Let $\phi \in D(\mathbb{R})$ be arbitrary. We have

$$\begin{aligned}\langle d_r \delta, \phi\rangle &= \frac{1}{r}\langle \delta, d_{1/r}\phi\rangle \\ &= \frac{1}{r}\phi(r\cdot 0)\ \ \text{(by definition of } \delta) \\ &= \frac{1}{r}\phi(0) \\ &= \langle \frac{1}{r}\delta, \phi\rangle\end{aligned}$$

Since ϕ is arbitrary, we conclude $d_r\delta = \frac{1}{r}\delta$ and hence δ is homogeneous of degree -1.

[c.] Let T' be the derivative of a distribution T. Let $\phi \in D(\mathbb{R})$ be arbitrary. Assume that T is homogeneous of degree n. We have

$$\begin{aligned}
\langle d_r T', \phi \rangle &= \frac{1}{r} \langle T', d_{1/r}\phi \rangle \\
&= -\frac{1}{r} \langle T, (d_{1/r}\phi)' \rangle \\
&= -\frac{1}{r} \langle T, \frac{1}{r} d_{1/r}(\phi') \rangle \\
&= -\frac{1}{r^2} \langle T, d_{1/r}(\phi') \rangle \\
&= -\frac{1}{r^2} \langle T, d_{1/r}\phi' \rangle \\
&= -\frac{1}{r} \langle d_r T, \phi' \rangle \\
&= -\frac{1}{r} \langle r^n T, \phi' \rangle \quad \text{(since } T \text{ has degree } n\text{)} \\
&= r^{n-1} \left(-\langle T, \phi' \rangle \right) \\
&= \langle r^{n-1} T', \phi \rangle.
\end{aligned}$$

Since ϕ is arbitrary, it follows $d_r T' = r^{n-1} T$ and hence T' is homogeneous of degree $n-1$.

8.0.46 Problem 6

Author: Mohamed Omar

Statement: Problem 6

Calculus of variations.

Proof:

Chapter 9

Winter 2009

Author: Ricky Kwok

9.0.47 Problem 1

Statement: Problem 1

Let $1 < p < 2$.

(a) Given an example of a function $f \in L^1(\mathbb{R})$ such that $f \notin L^p(\mathbb{R})$ and a function $g \in L^2(\mathbb{R})$ such that $g \notin L^p(\mathbb{R})$.

(b) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, prove that $f \in L^p(\mathbb{R})$.

Proof:
Let $f(x) = x^{-1/p}\chi_{(0,1)}$. Then

$$||f||_1 = \lim_{\varepsilon \to 0} \int_\varepsilon^1 x^{-1/p} \, \mathrm{d}x = \frac{x^{1-1/p}}{1-1/p}\Big|_0^1 = \frac{p}{p-1} < \infty.$$

However,

$$||f||_p^p = \lim_{\varepsilon \to 0} \int_\varepsilon^1 x^{-1} \, \mathrm{d}x = \lim_{\varepsilon \to 0} \ln(x)\Big|_\varepsilon^1 = -\lim_{\varepsilon \to 0} \log(\varepsilon) = \infty.$$

Hence $f \in L^1$, but $f \notin L^p$.

Let $g(x) = x^{-1/p}\chi_{(1,\infty)}$. Since $p < 2$, we have $1/p > 1/2$, or $2/p > 1$, so that $1 - 2/p < 0$.

$$||g||_2 = \int_1^\infty x^{-2/p} \, \mathrm{d}x = \frac{x^{1-2/p}}{1-2/p}\Big|_1^\infty = \lim_{x \to \infty} \frac{x^{1-2/p}}{1-2/p} - \frac{1}{1-2/p} = \frac{p}{p-2}.$$

However,

$$||g||_p^p = \int_1^\infty x^{-1} \, \mathrm{d}x = \infty.$$

Thus, $g \in L^2$, but $g \notin L^p$.

Now suppose $f \in L^1 \cap L^2$. Define

$$A = \{x \in \mathbb{R} : |f(x)| > 1\}.$$

Then we have

$$\begin{aligned} ||f||_p^p &= \int_{\mathbb{R}} |f(x)|^p \, \mathrm{d}x \\ &= \int_A |f(x)|^p \, \mathrm{d}x + \int_{A^c} |f(x)|^p \, \mathrm{d}x \\ &\leq \int_A |f(x)|^2 \, \mathrm{d}x + \int_{A^c} |f(x)|^1 \, \mathrm{d}x \leq ||f||_2^2 + ||f||_1^1 \end{aligned}$$

Therefore, if $f \in L^1$ and $f \in L^2$, then $f \in L^p(\mathbb{R})$. □

9.0.48 Problem 2

Statement: Problem 2

(a) State the Weierstrass approximation theorem.

(b) Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and

$$\int_0^1 x^n f(x)\ \mathrm{d}x = 0$$

for all nonnegative integers n. Prove that $f = 0$.

Proof:

The Weierstrass approximation theorem states that for a closed interval $[a, b]$, the space of continuous functions on this interval $C([a, b])$ can be uniformly approximated with polynomials, i.e. there exists a sequence $(f_m) \in \Pi([a, b])$ such that

$$\lim_{m\to\infty} ||f_m - f||_{unif} = \lim_{m\to\infty} \sup_{x\in[a,b]} |f_m(x) - f(x)| = 0,$$

where $\Pi([a, b])$ is the space of polynomials with domain $[a, b]$.

Since f is continuous over $[0, 1]$, we can take a sequence of polynomials $(f_m) : [0, 1] \to \mathbb{R}$ to approximate f uniformly. Then we can pass limit under the integral sign (see Exercise 2.2).

$$\lim_{m\to\infty} \int_0^1 |f_m(x)f(x) - f(x)f(x)|\ \mathrm{d}x \le \int_0^1 \lim_{m\to\infty} |f_m(x) - f(x)||f(x)|\ \mathrm{d}x \le \lim_{m\to\infty} ||f_m - f||_{unif} \int_0^1 |f(x)|$$

This shows $\int_0^1 f(x)^2\ \mathrm{d}x = 0$. Since $(f(x))^2 \ge 0$ for all x, and f is continuous, $f(x) = 0$ for all x, i.e. $f \equiv 0$. □

9.0.49 Problem 3

Statement: Problem 3

(a) Define strong convergence, $x_n \to x$, and weak convergence, $x_n \rightharpoonup x$, of a sequence (x_n) in a Hilbert space $\mathcal{H}$.

(b) If $x_n \rightharpoonup x$ weakly in $\mathcal{H}$ and $||x_n|| \to ||x||$, prove that $x_n \to x$ strongly.

(c) Give an example of a Hilbert space $\mathcal{H}$ and a sequence (x_n) in $\mathcal{H}$ such that $x_n \rightharpoonup x$ weakly and

$$||x|| < \liminf_{n\to\infty} ||x_n||.$$

Proof:

(a) Let $||\cdot||$ denote the norm in the Hilbert space. Strong convergence is

$$\lim_{n\to\infty} ||x_n - x|| = 0.$$

Let $(\cdot,\cdot)$ denote the inner product in the Hilbert space. Weak convergence is

$$\lim_{n\to\infty} (x_n, y) = (x, y) \qquad \forall y \in \mathcal{H}.$$

(b) Suppose $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$. Then $||x_n||^2 \to ||x||$ and $(x_n, x) \to (x, x) = ||x||^2$ and $(x, x_n) = \overline{(x_n, x)} \to \overline{(x, x)} = ||x||^2$ since $x \in \mathcal{H}$,

$$\begin{aligned}||x_n - x||^2 &= (x_n - x, x_n - x) = (x_n, x_n) - (x_n, x) - (x, x_n) + (x, x) \\ &\to ||x||^2 - ||x||^2 - ||x||^2 + ||x||^2 = 0.\end{aligned}$$

Hence, $||x_n - x||^2 \to 0$, therefore $||x_n - x|| \to 0$.

(c) Let $\mathcal{H} = L^2(\mathbb{R})$, be the set of all square-integrable functions on the real line. Define the sequence $g_n = \chi_{(n,n+1)}$ denote the characteristic function on the intervals $(n, n+1)$. Then we have for every $n \in \mathbb{N}$,

$$||g_n|| = \int_n^{n+1} 1^2 \, \mathrm{d}x = 1.$$

I claim $g_n \rightharpoonup 0$. To see this, let $f \in L^2(\mathbb{R})$. Then

$$||f||_2^2 = \int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x = \sum_{n=-\infty}^{\infty} \int_n^{n+1} |f(x)|^2 \, \mathrm{d}x.$$

Now, Bessel's inequality states that

$$\lim_{n\to\pm\infty}\int_n^{n+1} |f(x)|^2 \ \mathrm{d}x = 0$$

Then the inner product of g_n with f is given by

$$(g_n, f) = \int_n^{n+1} f(x) \ \mathrm{d}x =: y_n.$$

Notice that by Jensen's inequality,

$$\int_n^{n+1} |f(x)|^2 \ \mathrm{d}x \geq \left(\int_n^{n+1} f(x) \ \mathrm{d}x\right)^2.$$

So we have $\int_n^{n+1} f(x) \ \mathrm{d}x \to 0$ as $n \to \infty$. Therefore,

$$(g_n, f) \to 0 \qquad \Rightarrow \qquad g_n \rightharpoonup 0.$$

Then $||0|| = 0 < 1 = \liminf_{n\to\infty} ||g_n||$. Therefore $g_n = \chi_{(n,n+1)}$ satisfies the weak convergence to 0 and $||0|| < \liminf_{n\to\infty} ||g_n|| = 1$. □

9.0.50 Problem 4

Statement: Problem 4

Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a complex Hilbert space $\mathcal{H}$ such that

$$T^* = -T, \qquad T^2 = -I$$

and $T \neq \pm iI$. Define

$$P = \frac{1}{2}(I + iT), \qquad Q = \frac{1}{2}(I - iT).$$

(a) Prove that P, Q are orthogonal projections on $\mathcal{H}$.

(b) Determine the spectrum of T, and classify it.

Proof:

(a) Notice

$$P^2 = \frac{1}{4}(I + iT)(I + iT) = \frac{1}{4}(I + iT + iT + i^2T^2) = \frac{1}{4}(2I + 2iT) = P,$$

and similarly $Q^2 = Q$. Then for all elements $y \in \mathcal{H}$,

$$\begin{aligned}(Px, y) &= \left(\frac{x}{2} + \frac{iTx}{2}, y\right) = \frac{1}{2}(x, y) + \frac{-i}{2}(Tx, y) \\ &= \left(x, \frac{y}{2}\right) + \frac{-i}{2}(x, T^*y) = \left(x, \frac{y}{2} + \frac{-i}{2}T^*y\right) = \left(x, \left(\frac{1}{2}I + \frac{1}{2}iT\right)y\right) = (x, Py),\end{aligned}$$

and similarly $(Qx, y) = (x, Qy)$. Therefore P and Q are orthogonal projections (projections and self-adjoint).

(b) To find the spectrum of $T, \sigma(T)$, we first find the resolvent of $T, \rho(T)$ and then $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T : \mathcal{H} \to \mathcal{H}, \text{ is one-to-one and onto}\}.$$

We use the fact that an operator is one-to-one and onto if and only if it is invertible. So, we must find all values λ such that $(\lambda I - T)^{-1}$ is well-defined. Using $T^2 = -I$, we take a guess that $(\lambda I - T)^{-1} = c(\lambda I + T)$ for some $c \in \mathbb{C}$. To solve for c,

$$I = (\lambda I - T)c(\lambda I + T) = c(\lambda^2 I - T^2) = c(\lambda^2 + 1)I.$$

We can see that $c = \frac{1}{\lambda^2+1}$. Hence,

$$(\lambda I - T)^{-1} = \frac{\lambda I + T}{1 + \lambda^2}.$$

This operator is well-defined for all values $\lambda \in \mathbb{C} \setminus \{\pm i\}$. However, since $T \neq \pm iI$, $\pm i \notin \sigma(T)$, the resolvent set is consists of all complex numbers $\rho(T) = \mathbb{C}$. Therefore, $\sigma(T) = \emptyset$. $\square$

9.0.51 Problem 5

Statement: Problem 5

Let $S(\mathbb{R})$ be the Schwartz space of smooth, rapidly decreasing functions $f : \mathbb{R} \to \mathbb{C}$. Define an operator $H : S(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(\widehat{Hf})(\xi) = \text{i}\operatorname{sgn}(\xi)\widehat{f}(\xi) = \begin{cases} \text{i}\widehat{f}(\xi) & \text{if } \xi > 0 \\ -\text{i}\widehat{f}(\xi) & \text{if } \xi < 0, \end{cases}$$

where $\widehat{f}$ denotes the Fourier transform of f.

(a) Why is $Hf \in L^2(\mathbb{R})$ for any $f \in S(\mathbb{R})$?

(b) If $f \in (\mathbb{R})$ and $Hf \in L^1(\mathbb{R})$, show that

$$\int_{\mathbb{R}} f(x)\ \mathrm{d}x = 0.$$

[Hint: you may want to use the Riemann-Lebesgue Lemma.]

Proof:

(a) The function f is in any L^p space since f is in the Schwartz space. In particular, $f \in L^1 \cap L^2$, implying the Fourier transform of f is in L^2 and Plancherel's theorem holds:

$$||\widehat{f}||_2 = ||f||_2.$$

Notice that the L^2 norm of $\widehat{Hf}$ is equal to the L^2 norm of $\widehat{f}$ because

$$\left|\widehat{Hf}(\xi)\right| = \left|\widehat{f}(\xi)\right| \qquad \forall \xi \in \mathbb{R}.$$

Combining these two, we have

$$||Hf||_2 = ||\widehat{Hf}||_2 = ||\widehat{f}||_2 = ||f||_2 < \infty.$$

Therefore, $Hf \in L^2$.

(b) Since $Hf \in L^1$, then the Riemann-Lebesgue Lemma states that $\widehat{Hf}$ is continuous and $\widehat{Hf}(\xi) \to 0$ as $|\xi| \to \infty$. In particular $\widehat{Hf}$ is continuous at 0, so

$$\text{i} \cdot \lim_{\xi \to 0^+} \widehat{f}(\xi) = \lim_{\xi \to 0^+} \widehat{Hf}(\xi) = \lim_{\xi \to 0^-} \widehat{Hf}(\xi) = \text{-i} \cdot \lim_{\xi \to 0^-} \widehat{f}(\xi).$$

Since the left hand side is equal to the negative of the right hand side, these two must both equal zero. Hence, $\widehat{f}(0) = 0$. But by the definition of the Fourier transform,

$$0 = \widehat{f}(0) = \int_{\mathbb{R}} e^{2\pi i 0 x} f(x) \, \mathrm{d}x = \int_{\mathbb{R}} f(x) \, \mathrm{d}x.$$

Therefore f has zero integral. □

9.0.52 Problem 6

Statement: Problem 6

Let Δ denote the Laplace operator in $\mathbb{R}^3$.

(a) Prove that

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon^c}\frac{1}{|\mathbf{x}|}\Delta f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 4\pi f(0), \qquad \forall f\in S(\mathbb{R}^3)$$

where B_ε^c is the complement of the ball of radius ε centered at the origin.

(b) Find the solution u of the Poisson problem

$$\Delta u = 4\pi f(\mathbf{x}), \qquad \lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0$$

for $f\in S(\mathbb{R}^3)$.

Proof:

Chapter 10

Fall 2009

10.0.53 Problem 1

Author: Owen Lewis

Statement: Problem Number 1

For $\epsilon > 0$, let η_ϵ denote the family of *standard mollfiers* on $\mathbb{R}^2$. Given $u \in L^2(\mathbb{R}^2)$, define the function

$$u_\epsilon = \eta_\epsilon * u \text{ in } \mathbb{R}^2.$$

Prove that

$$\epsilon \|Du_\epsilon\|_{L^2} \leq C\|u\|_{L^2},$$

where the constant C depends on the mollifier, not on u.

Proof:
By definition, we have that

$$u_\epsilon(x) = \eta_\epsilon * u = \int_{\mathbb{R}^2} \eta_\epsilon(x-y)u(y)dy.$$

Taking the partial derivative with respect to x_i, and utilizing the Lebesgue Dominated Convergence theorem yields

$$\partial_{x_i} u_\epsilon(x) = \partial_{x_i} \int_{\mathbb{R}^2} \eta_\epsilon(x-y)u(y)dy = \int_{\mathbb{R}^2} \partial_{x_i}\eta_\epsilon(x-y)u(y)dy.$$

Now, by Young's inequality for convolutions (letting $p = 2$, $q = 1$ and $r = 2$) we have

$$\|\partial_{x_i} u_\epsilon\|_{L^2} \leq \|\partial_{x_i}\eta_\epsilon\|_{L^1}\|u\|_{L^2}. \tag{10.1}$$

Now, examining $\|\partial_{x_i}\eta_\epsilon\|_{L^1}$, we have

$$\begin{aligned}
\|\partial_{x_i}\eta_\epsilon\|_{L^1} &= \int_{\mathbb{R}^2} |\partial_{x_i}\eta_\epsilon(x)|dx \\
&= \int_{\mathbb{R}^2} \left|\frac{1}{\epsilon^2}\partial_{x_i}\eta\left(\frac{x}{\epsilon}\right)\right| dx.
\end{aligned}$$

Making the change of variables $y = \frac{x}{\epsilon}$, we have that $\partial_{x_i} = \frac{1}{\epsilon}\partial_{x_1}$ and $dx = \epsilon^2 dy$. Therefore the above expression becomes

$$\frac{1}{\epsilon}\int_{\mathbb{R}^2} |\partial_{y_i}\eta\,(y)|\, dy.$$

Because $\eta \in C_c^\infty(\mathbb{R}^2)$, there exists a constant C that bounds the above expression. Thus we have

$$\|\partial_{x_i} u_\epsilon\|_{L^2} \leq \frac{C}{\epsilon}.$$

Combining this with (10.1), and noting that i is arbitrary gives the result

$$\epsilon\|Du_\epsilon\|_{L^2} \leq C\|u\|_{L^2}.$$

This is exactly what we wanted to show. □

10.0.54 Problem 2

Author: Owen Lewis

Statement: Problem Number

Let $B(0,1) \subset \mathbb{R}^3$ denote the unit ball $\{|x| < 1\}$. Prove that $u = \log|x| \in H^1(B(0,1))$.

Proof:
We begin by showing that $u \in L^2(B(0,1))$. We first note that for $|x| < 1$, then $|\log(|x|)| \le \frac{1}{|x|}$. Therefore, utilizing polar coordinates we have

$$\begin{aligned}
\|u\|_{L^2(B(0,1))} &= \int_{B(0,1)} u^2 dx \\
&= \int_{B(0,1)} \log^2(|x|) dx \\
&\le \int_{B(0,1)} \frac{1}{|x|^2} dx \\
&= C \int_0^1 \frac{1}{r^2} r^2 dr \\
&= C.
\end{aligned}$$

Therefore, $u \in L^2(B(0,1))$. Now a quick calculation shows that

$$\partial_{x_i} u = \frac{1}{|x|} \frac{x_i}{|x|},$$

and thus

$$Du = \frac{x}{|x|^2}.$$

By definition,

$$\begin{aligned}
\|Du\|_{L^2 L^2(B(0,1))} &= \int_{B(0,1)} Du \cdot Du \, dx \\
&= \int_{B(0,1)} \frac{x \cdot x}{|x|^4} dx \\
&= C \int_0^1 \frac{r^2}{r^4} r^2 dr \\
&= C.
\end{aligned}$$

Therefore $Du \in L^2(B(0,1))$. Combined with the previous result we have that $u \in H^1(B(0,1))$.
This is exactly what we wanted to show. □

10.0.55 Problem 6

Author: Owen Lewis

Statement: Problem Number

Prove that a normed linear space is complete if and only if every absolutely summable sequence is summable.

Proof:
First, we assume that our space is complete. Let $\{x_n\}$ be an absolutely summable sequence. Now define $y_m = \sum_{n=1}^{m} x_n$. We have that

$$\|y_m - y_k\| = \| \sum_{n=m}^{k} x_n\| \leq \sum_{n=m}^{k} \|x_n\|.$$

Because $\{x_n\}$ is absolutely summable, we know that $\sum_{n=1}^{\infty} \|x_n\| = M$, and thus for any $\epsilon > 0$, we can choose m sufficiently large such that

$$\sum_{n=1}^{m} x_n < \epsilon.$$

Combined with the previous inequality, this shows that the sequence $\{y_m\}$ is Cauchy. By completeness, it has a limit x. Therefore we have

$$\sum_{n=1}^{\infty} x_n = \lim_{m\to\infty} \sum_{n=1}^{m} x_n = \lim_{m\to\infty} y_m = x.$$

This shows that every absolutely summable sequence is summable.

Now assume that every absolutely summable sequence is summable. Let $\{x_n\}$ be a Cauchy sequence. By passing to a subsequence if necessary, we can claim that $\|x_k - x_{k-1}\| \leq 2^{-k}$. Now define $y_1 = x_1$, and $y_k = x_k - x_{k-1}$ for $k > 1$. By definition,

$$\sum_{k=1}^{m} y_k = x_m.$$

Furthermore, by construction

$$\sum_{k=1}^{\infty} \|y_k\| = \sum_{k=1}^{\infty} \|x_k - x_{k-1}\| \leq \sum_{k=1}^{\infty} 2^{-k} < \infty.$$

Thus the sequence $\{y_k\}$ is absolutely summable. By assumption it is therefore summable, with limit x. Therefore,

$$\lim_{k\to\infty} x_k = \sum_{k=1}^{\infty} y_k = x,$$

which is to say that the *subsequence* $\{x_k\}$ is convergent. However, if any Cauchy sequence has a convergent subsequence, then the entire sequence is convergent. Thus, our original Cauchy sequence is convergent, and $x_n \to x$. This shows that the space is complete, and completes the proof.

Chapter 11

Winter 2010

Author: Nathan Hannon

11.0.56 Problem 1

Statement: Problem 1

Let (X, d) be a complete metric space, $\bar{x} \in X$ and $r > 0$. Set $D := \{x \in X : d(x, \bar{x}) \leq r\}$, and let $f : D \to X$ satisfying

$$d(f(x), f(y)) \leq k\, d(x, y)$$

for any $x, y \in D$, where $k \in (0, 1)$ is a constant.

Prove that if $d(\bar{x}, f(\bar{x}) \leq r(1-k)$ then f admits a unique fixed point.

(Guidelines: Assume the Banach fixed point theorem, also known as the contraction mapping theorem.)

Proof:
Let $x \in D$. Then

$$\begin{aligned} d(f(x), \bar{x}) &\leq d(f(x), f(\bar{x})) + d(f(\bar{x}), \bar{x}) \\ &\leq k\, d(x, \bar{x}) + r(1-k) \\ &\leq rk + r(1-k) \\ &= r. \end{aligned}$$

Hence $f(x) \in D$. Since x was arbitrary, we may write $f : D \to D$, and f satisfies the hypotheses of the Banach fixed point theorem. We conclude that f admits a unique fixed point. □

11.0.57 Problem 2

Statement: Problem 2

Give an example of two normed vector spaces, X and Y, and of a sequence of operators, $\{T_n\}_{n=0}^{\infty}$, $T_n \in L(X,Y)$ ($L(X,Y)$ is the space of the continuous operators from X to Y, with the topology induced by the operator norm) such that $\{T_n\}_{n=0}^{\infty}$ is a Cauchy sequence but it does not converge in $L(X,Y)$.

(Notice that Y cannot be a Banach space otherwise $L(X,Y)$ is complete.)

Proof:
Let $X = \mathbb{R}$ and let $Y \subset L^1([-1,1])$ be the subspace consisting of continuous functions. We note that any operator $T \in L(X,Y)$ is determined completely by $T(1)$ and that $\|T\| = \|T(1)\|$. It therefore suffices to find a Cauchy sequence f_n in Y that does not converge in Y.

Let $f_n = x^{\frac{1}{2n+1}} \in Y$. Then $f_n \to \operatorname{sgn} x$ in $L^1([-1,1])$. Hence (f_n) is Cauchy in $L^1([-1,1])$ and hence in Y. However, because $\operatorname{sgn} x \notin Y$, (f_n) does not converge in Y. □

11.0.58 Problem 3

Statement: Problem 3

Let (a_n) be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} a_n^3$$

converges. Show that

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

also converges.

Proof:
We note that $(a_n) \in \ell^3(\mathbb{N})$. Furthermore, since

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} < \infty,$$

we have $\left(\frac{1}{n}\right) \in \ell^{\frac{3}{2}}(\mathbb{N})$. By Hölder's inequality,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_n}{n} &\leq \|(a_n)\|_3 \left\| \left(\frac{1}{n} \right) \right\|_{\frac{3}{2}} \\ &< \infty. \end{aligned}$$

This is exactly what we wanted to show. □

11.0.59 Problem 4

Statement: Problem 4

> Suppose that $h : [0,1]^2 \to [0,1]^2$ is a continuously differentiable funcion from the square to the square with a continuously differentiable inverse h^{-1}. Define an operator T on the Hilbert space $L^2([0,1]^2)$ by the formula $T(f) = f \circ h$. Prove that T is a well-defined bounded operator on this Hilbert space.

Proof:
Let $u = h(x)$ and let $J(u)$ denote the Jacobian of x with respect to u. Since h and h^{-1} are continuously differentiable, J is bounded. Let

$$M = \sup_{u \in [0,1]^2} |J(u)|.$$

We also note that h maps $[0,1]^2$ to itself. We can use a change of variables to evaluate $\|T(f)\|_2$:

$$\begin{aligned} \|T(f)\|_2^2 &= \int_{[0,1]^2} |(f \circ h)(x)|^2 \, dx \\ &= \int_{[0,1]^2} |J(u)| \, |f(u)|^2 \, du \\ &\leq \int_{[0,1]^2} M |f(u)|^2 \, du \\ &= M \|f\|_2^2. \end{aligned}$$

Hence

$$\|T(f)\|_2 \leq \sqrt{M} \|f\|_2;$$

i. e., T is well-defined and bounded. □

11.0.60 Problem 5

Statement: Problem 5

Let $H^s(\mathbb{R})$ denote the Sobolev space of order s on the real line $\mathbb{R}$, and let

$$\|u\|_s = \left(\int_{\mathbb{R}} \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

denote the norm on $H^s(\mathbb{R})$, where $\hat{u}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} u(x) e^{-ix\xi} \, dx$ denotes the Fourier transform of u.

Suppose that $r < s < t$, all real, and $\epsilon > 0$ is given. Show that there exists a constant $C > 0$ such that

$$\|u\|_s \leq \epsilon \|u\|_t + C\|u\|_r \;\; \forall u \in H^t(\mathbb{R}).$$

Proof:
Let

$$g(\xi) = \left(1 + |\xi|^2\right)^{\frac{1}{2}}.$$

We note that g is strictly positive.

Suppose that $u \in H^t(\mathbb{R})$ and define a measure ν on $\mathbb{R}$ by

$$d\nu = |\hat{u}(\xi)|^2 \, d\xi.$$

We note that

$$\begin{aligned} \|g^r\|_{L^2(\mathbb{R},\nu)} &= \left(\int_{\mathbb{R}} \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \\ &= \|u\|_r. \end{aligned}$$

Similarly, $\|g^s\|_{L^2(\mathbb{R},\nu)} = \|u\|_s$ and $\|g^t\|_{L^2(\mathbb{R},\nu)} = \|u\|_t$.

Let

$$C = \epsilon^{\frac{s-r}{s-t}}.$$

We note that C does not depend on the choice of u.

Suppose that $g(\xi) = y$. If $y \geq \epsilon^{\frac{1}{s-t}}$, then

$$\begin{aligned} y^s &= y^{s-t} y^t \\ &\leq \epsilon y^t \\ &\leq \epsilon y^t + C y^r. \end{aligned}$$

Otherwise, $y < \epsilon^{\frac{1}{s-t}}$, and

$$\begin{aligned} y^s &= y^{s-r} y^r \\ &< C y^r \\ &\leq \epsilon y^t + C y^r. \end{aligned}$$

It follows that

$$g(\xi)^s \leq \epsilon g(\xi)^t + C g(\xi)^r \;\; \forall \xi \in \mathbb{R}.$$

Using Minkowski's inequality,

$$
\begin{aligned}
\|g^s\|_{L^2(\mathbb{R},\nu)} &\leq \|\epsilon g^t + C g^r\|_{L^2(\mathbb{R},\nu)} \\
&\leq \epsilon\|g^t\|_{L^2(\mathbb{R},\nu)} + C\|g^r\|_{L^2(\mathbb{R},\nu)} \\
\|u\|_s &\leq \epsilon\|u\|_t + C\|u\|_r.
\end{aligned}
$$

This is exactly what we wanted to show. □

11.0.61 Problem 6

Statement: Problem 6

Let $f : [0, 1] \to \mathbb{R}$. Show that f is continuous if and only if the graph of f is compact in $\mathbb{R}^2$.

Proof:
Denote by G the graph of f in $\mathbb{R}^2$. Suppose that f is continuous. Since $[0, 1]$ is compact, f must be bounded. Hence G is bounded.

Now suppose that $(x_n, y_n) \in G$ and $(x_n, y_n) \to (x, y)$. Then $x_n \to x$ and $y_n \to y$. However, because f is continuous, $y_n = f(x_n) \to f(x)$. It follows that $y = f(x)$ and that $(x, y) \in G$. Since this holds for all convergent sequences (x_n, y_n), G is closed.

By the Heine-Borel theorem, G is compact.

Conversely, suppose that G is compact. By the Heine-Borel theorem, G is closed and bounded. Suppose that $x_n \to x$ and that $f(x_n) = y_n$. Let $y = \liminf y_n$ and $y' = \limsup y_n$. Since G is bounded,
$-\infty < y < y' < \infty$. Choose subsequences y_{n_k} and $y_{n'_k}$ such that $y_{n_k} \to y$ and $y_{n'_k} \to y'$.

Then $(x_{n_k}, y_{n_k}) \to (x, y)$ and $(x_{n'_k}, y_{n'_k}) \to (x, y')$. Since G is closed, $(x, y), (x, y') \in G$, and it follows that $y = y'$. We conclude that $f(x_n) \to y = f(x)$.

Because this holds for all convergent sequences (x_n), f is continuous.

Therefore, f is continuous if and only if G is compact. □

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