Old Prelim Solutions

By Jeff Irion and Ryan Halabi

Last Update: August 3, 2012

Contents

September 2007

Problem 1.1

Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x^n)\ dx$$

exists and evaluate the limit. Does the limit always exist if f is only assumed to be integrable?

$|f(x^n)| \leq M$ where M is the bound for f which exists due to its continuity. By the dominated convergence theorem this allows us to bring in the limit

$$\lim_{n\to\infty} \int_0^1 f(x^n)\ dx = \int_0^1 \lim_{n\to\infty} f(x^n)\ dx = \int_0^1 f(\lim_{n\to\infty} x^n)\ dx = \int_0^1 f(0)\ dx$$

If f is just integrable but not continuous, such as $f(x) = x^{-1/2}$ then we have

$$\begin{aligned} \lim \int x^{-n/2}\ dx &= \lim \frac{x^{-n/2+1}}{-n/2+1} \\ &= \infty \end{aligned}$$

so the limit does not exist.

Problem 1.2

Define

$$T(t)\left(\sum f_n e^{inx}\right) = \sum e^{iw_n t} f_n e^{inx}$$

(a)

Show that $T(t)$ is a unitary map.

$$\begin{aligned}\langle T(t)f, T(t)g\rangle &= \int \sum e^{-iw_n t}\overline{f_n}e^{-inx} \sum e^{iw_n t} g_n e^{inx}\ dx \\ &= \int \overline{f_n} g_n\ dx \\ &= \langle f, g\rangle\end{aligned}$$

(b)

Show that $T(s)T(t) = T(s+t)$.

$$\begin{aligned}T(s)T(t) &= T(s)\sum e^{iw_n t} f_n e^{inx} \\ &= \sum e^{iw_n s} e^{iw_n t} f_n e^{inx} \\ &= \sum e^{iw_n (t+s+} f_n e^{inx} \\ &= T(s+t)\end{aligned}$$

(c)

Show that if $f \in C^\infty$ then so is $T(t)f$.

January 2008

Problem 2.1

Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = (-1)^n x^n (1-x)$$

(a)

Show that $\sum f_n$ converges uniformly on $[0,1]$

We are looking at the summation $\sum^N (-1)^n x^n (1-x) = (1-x)\sum^N -x^n = (1-x)\frac{-x^N}{1+x} = 1$ so f_n converges uniformly to 1.

(b)

Show that $\sum |f_n|$ converges pointwise on $[0,1]$ but not uniformly

For $x \neq 1$, $\sum^N x^n(1-x) = (1-x)\sum^N x^n = \frac{1-x}{1-x}x^N = x^N$ which converges pointwise to 0.

Problem 2.2

Consider $X = \mathbb{R}^2$ equipped with the Euclidean metric,

$$e(x, y) = [(x^1 - y^1)^2 + (x^2 - y^2)]^{1/2},$$

where $x = (x_1, x_2) \in \mathbb{R}^2, y = (y_1, y_2) \in \mathbb{R}^2$. Define $d : X \times X \to \mathbb{R}$ by

$$\begin{aligned} d(x, y) &= e(x, y) \quad \text{when } x, y \text{ lie on the same ray through the origin} \\ &= e(x, 0) + e(0, y) \quad \text{if not} \end{aligned}$$

(a)

Prove that (X, d) is a metric space.

$d(x, y) = 0$ is only possible if x, y lie on the same ray, and only then possible if $x = y$

$d(x, y) = d(y, x)$ is trivial.

$d(x, z) \geq d(x, y) + d(y, z)$ is more difficult, if x, z are on the same ray this is trivial.

If x, z are not on the same ray then $d(x, z) = e(x, 0) + e(0, y)$. We have two choices for y.
If y is not on the same ray as either of x, z we have $d(x, y) + d(y, z) = e(x, 0) + e(0, y) + e(y, 0) + e(0, z) > e(x, 0) + e(0, z)$
If y is on the same ray as (at most) one of x, z we have $d(x, y) + d(y, z) = e(x, 0) + e(0, y) + e(y, z) > e(x, 0) + e(0, z)$
and so we are done showing d is a metric.

(b)

Give an example of a set that is open in (X, d) but not open in (X, e).

Take the open unit ball $B_1((100, 0))$, under the d metric this is a line on x-axis. However under the usual e metric this line is not open since it is a line in a 2 dimensional space.

Problem 2.3

Prove the Hahn-Banach Theorem

Problem 2.4

(a)

Show that the Resolvent is open.

Suppose we have λ_0 in the resolvent, what restrictions do we have on λ for $\lambda I - A$ to be invertible?

$$\lambda I - A = (\lambda_0 I - A)(I - (\lambda_0 - \lambda)(\lambda_0 I - A)^{-1})$$

we see that only way this operator is invertible is if $\|(\lambda_0 - \lambda)((\lambda_0 I - A)^{-1})\| < 1$ so that $|(\lambda_0 - \lambda)| < \frac{1}{\|((\lambda_0 I - A)^{-1})\|}$

(b)

Prove that $\|\lambda_0 I - A)^{-1}\| \geq \frac{1}{d(\lambda_0, \sigma(A))}$ where d is the inf distance between λ_0 an element of the spectrum.

Well if it wasn't larger then that, then this would imply there was an element of the spectrum for which $((\sigma I - A)^{-1})$ was invertible which is clearly not possible.

September 2008

Problem 3.1

Prove that the dual space of c_0 is ℓ^1, where

$$c_0 = \{x = (x_n) \text{ such that } \lim x_n = 0\}$$

Choose some linear function ϕ, then $\phi : c_0 \to \mathbb{R}$, we can learn alot about ϕ but looking at how it interacts elements of the form $e_i = (0, \dots, 1, 0, \dots)$, $\phi(e_i) = \lambda_i$, so we can actually think of ϕ as looking something like $\phi(x) = \sum_{i=0}^{\infty} x_i \phi(e_i)$ so what constraints do we need on λ_i to ensure that $\sum_{i=0}^{\infty} x_i \lambda_i$ converges?

So all we know is that at some point the absolute value of the elements of the tail of c_0 have to be less then ϵ

so what constraints do we have on λ_i?

$$\sum_{i=N}^{\infty} |x_i| \lambda_i \leq \sum_{i=N}^{\infty} \epsilon \lambda_i$$

This means that we need the sum of λ_i to converge, however we actually need the sum of the absolute value of λ_i to converge as well.

Take some sequence x, multiply each element of x by $\operatorname{sgn}(\phi(e_i)x_i)$, this means when you apply ϕ to x you have $\sum x_i \operatorname{sgn}(\phi(e_i)x_i)\phi(e_i)$ which is equal to $\sum \epsilon |\phi(e_i)| = \sum \epsilon |\lambda_i|$

This shows that $\{lambda_i\} \in \ell^1$ Choose a sequence in ℓ^1, can we make a linear function on c_0 out of this? Using the fact that c_0 is a closed banach subspace of ℓ^∞ we use the sup norm onf c_0.

choose $x \in c_0$, $y \in \ell^1$, do the same pairing that we had above

$$\begin{aligned} |\sum x_i y_i| &\leq \sup |x_i| \sum |y_i| \\ &\leq C \|x\|_{supnorm} \end{aligned}$$

Problem 3.2

Let $\{f_n\}$ be a sequence of differentiable functions on a finite interval $[a, b]$ such that the functions themselves and their derivatives are uniformly bounded on $[a, b]$. Prove that $\{f_n\}$ has a uniformly converging subsequence.

Arzelà-Ascoli states that a sequence of continuous functions has a uniformly convergent subsequence if the sequence is uniformly bounded and equicontinuous.

We are given that uniform boundedness is satisfied, to show equicontinuity we use the mean value theorem

$$\begin{aligned}|f(x) - f(y)| &= |f'(a)||x - y| \\ &\leq C|x - y|\end{aligned}$$

where the C is the bound for the functions and derivatives given. Arzela Ascoli is satisfied and so we are done.

Problem 3.3

Let $f \in L^1(\mathbb{R})$ and V_f be the closed subspace generated by the translates of f, i.e. $V_f := \{f(\cdot - y) \ \forall\, y \in \mathbb{R}$. Suppose that $\hat{f}(\xi_0) = 0$ for some ξ_0. Show that $\hat{h}(\xi_0) = 0$ for all $h \in V_f$. Show that if $V_f = L^1(\mathbb{R})$, then $\hat{f}$ never vanishes.

Source: `http://math.stackexchange.com/questions/143118/is-it-true-that-a-fourier-transform-of-f-never-v`

Assume by contradiction that

$$\widehat{f}(\psi) = 0.$$

Then, if h is a translate of f it is easy to show that

$$\widehat{h}(\psi) = 0.$$

Now, let $g \in V_f$. Then $g = \lim h_n$ where h_n are linear combinations of translates of f. Thus

$$\widehat{h_n}(\psi) = 0,$$

and since $h_n \to g$ in L^1 we get $\widehat{h_n}(\psi) \to \widehat{g}(\psi)$.

Thus, we indeed get

$$\widehat{g}(\psi) = 0 \qquad \forall\, g \in V_f.$$

Now to get the contradiction, we need to use the fact that for each $\psi \in \widehat{G}$ there exists some $g \in L^1$ so that $\widehat{g}(\psi) \neq 0$.

This is easy, let u be any compactly supported continuous function on $\hat{G}$ so that $u(\psi) \neq 0$. Then $g = u \check{*} \tilde{u}$ works, where $\check{}$ is the inverse Fourier Transform. This last part can probably be proven much easier, for example if g is non-zero, then $\hat{g}$ is not vanishing at some point, and then by multiplying g by the right character $\widehat{e^{\cdots} g}$ is not vanishing at ψ.

I don't entirely understand why you take the convolution of u with its conjugate, would you explain this?

Because $\check{u}$ is not necesarily an L^1 function, but it is an L^2. Then $|\check{u}|^2$ is L^1, and $|\check{u}|^2 = u \check{*} \tilde{u}$.

Problem 3.4

(a)

State the Stone-Weierstrass theorem for a compact Hausdorff space X.

(b)

Prove that the algebra generated by functions of the form $f(x, y) = g(x)h(y)$ where $g, h \in C(X)$ is dense in $C(X \times X)$.

Problem 3.5

For $r > 0$, define the dilation $d_r f : \mathbb{R} \to \mathbb{R}$ of a function $f : \mathbb{R} \to \mathbb{R}$ by $d_r f(x) = f(rx)$, and the dilation $d_r T$ of a distribution $T \in \mathcal{D}(\mathbb{R})$ by

$$\langle d_r T, \phi \rangle = \frac{1}{r} \langle T, D_{1/r}\phi \rangle \quad \text{for all test functions } \phi \in \mathcal{D}(\mathbb{R})$$

(a)

Show that the dilation of a regular distribution T_f given by

$$\langle T_f, \phi \rangle = \int f(x)\phi(x)\ dx$$

agrees with the dilation of the corresponding function f.

Let's take the dilation of T_f

$$\begin{aligned}
\langle d_r T, \phi \rangle &= \frac{1}{r} \langle T, d_{1/r}\phi \rangle \\
&= \frac{1}{r} \int f(x)\phi(x/r)\ dx \\
&= \frac{1}{r} \int f(ry)\phi(y) r\ dy \\
&= \int f(ry)\phi(y)\ dy \\
&= \int d_r f(y)\phi(y)\ dy
\end{aligned}$$

we see that the dilation of T_f simply dilates f

(b)

A distribution is homogeneous of degree n if $d_r T = r^n T$. Show that the δ-distribution is homogeneous of degree -1.

$$\begin{aligned}
\langle d_r \delta, \phi \rangle &= \frac{1}{r} \langle \delta, d_{1/r}\phi \rangle \\
&= \frac{1}{r}\phi(0) \\
&= \langle \frac{1}{r}\delta, \phi \rangle
\end{aligned}$$

so we see that $d_r \delta = r^{-1}\delta$.

(c)

If T is a homogeneous distribution of degree n, prove that the derivative T' is a homogeneous distribution of degree $n - 1$.

We look at T',

$$\begin{aligned}
\langle d_r T', \phi \rangle &= \frac{1}{r} \langle T', d_{1/r}\phi \rangle \\
&= -\frac{1}{r} \langle T, (d_{1/r}\phi)' \rangle \\
&= -\frac{1}{r^2} \langle T, \phi'(x/r) \rangle \\
&= -\frac{1}{r^2} \langle T, d_{1/r}\phi' \rangle \\
&= -\frac{1}{r} \langle d_r T, \phi' \rangle \\
&= -\frac{1}{r} \langle r^n T, \phi' \rangle \\
&= \frac{1}{r} \langle r^n T', \phi \rangle \\
&= \langle r^{n-1} T', \phi \rangle
\end{aligned}$$

Problem 3.6

Let $\ell^2(\mathbb{N})$ be the space of square-summable, real sequences $x = (x_1, x_2, x_3, \dots)$ with norm

$$\|x\| = \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2}$$

Define $F : \ell^2(\mathbb{N}) \to \mathbb{R}$ by

$$F(x) = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} x_n^2 - x_n^4 \right\}$$

(a)

Prove that F is differentiable at $x = 0$, with derivative $F'(0) : \ell^2(\mathbb{N}) \to \mathbb{R}$ equal to zero.

$$\frac{F(x+h) - F(x)}{h}$$

(b)

Show that the second derivative of F at $x = 0$

$$F''(0) : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \mathbb{R}$$

is positive definite meaning that

$$F''(0)(h, h) > 0$$

for every $h \in \ell^2(\mathbb{N})$.

(c)

Show that F does not attain at local minimum at $x = 0$.

January 2009

Problem 4.1

Let $1 < p < 2$.

(a)

Give an example of a function $f \in L^1(\mathbb{R})$ such that $f \notin L^p(\mathbb{R})$ and a function $g \in L^2(\mathbb{R})$ such that $g \notin L^p(\mathbb{R})$.

Choose the function $f = \frac{1}{x}\chi_{[1,\infty]}$

$$\begin{aligned}\int \frac{1}{x}\, dx &= \log(x)\Big|_1^\infty \\ &= \infty\end{aligned}$$

so $f \notin L^1(\mathbb{R})$

$$\begin{aligned}\int \frac{1}{x^2}\, dx &= -\frac{1}{x}\Big|_1^\infty \\ &= 0 + 1 \\ &= 1\end{aligned}$$

so that $f \in L^2(\mathbb{R})$

Now look at the function $g(x) = x^{-\frac{1}{2}}\chi_{(0,1)}$

$$\begin{aligned}\int x^{-\frac{1}{2}}\, dx &= 2x^{\frac{1}{2}}\Big|_0^1 \\ &= 2\end{aligned}$$

so $g \in L^1(\mathbb{R})$

$$\begin{aligned}\int x^{-1}\, dx &= \log(x)\Big|_0^1 \\ &= 0 + \infty \\ &= \infty\end{aligned}$$

so that $g \notin L^2(\mathbb{R})$

(b)

If $f \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$, prove that $f \in L^p(\mathbb{R})$.

There exists some $0 < \lambda < 1$ such that $\frac{\lambda}{1} + \frac{1-\lambda}{2} = \frac{1}{p}$ for $1 < p < 2$, rewriting this we have

$$\frac{\lambda p}{1} + \frac{(1-\lambda)p}{2} = 1$$

our proof will appeal to Holder's Inequality which gives us the intuition for the steps below

$$
\begin{aligned}
\|f\|_{L^p(\mathbb{R})} &= \left(\int |f|^p\right)^{\frac{1}{p}} \\
&= \left(\int |f|^{\lambda p}|f|^{(1-\lambda)p}\right)^{\frac{1}{p}} \\
&\leq \left(\left(\int |f|^{\lambda p \frac{1}{\lambda p}}\right)^{\lambda p} \left(\int |f|^{(1-\lambda)p\frac{2}{(1-\lambda)p}}\right)^{\frac{(1-\lambda)p}{2}}\right)^{\frac{1}{p}} \\
&\leq \left(\int |f|\right)^{\lambda} \left(\int |f|^2\right)^{\frac{(1-\lambda)}{2}} \\
&\leq \|f\|_{L^1\mathbb{R}}^{\lambda} \|f\|_{L^2(\mathbb{R})}^{\frac{(1-\lambda)}{2}}
\end{aligned}
$$

Problem 4.2

(a)

State the Weierstrass approximation theorem.

Suppose f is a continuous complex-valued function defined on the real interval $[a, b]$. For every $\epsilon > 0$, there exists a polynomial function p over $\mathbb{C}$ such that for all $x \in [a, b]$, we have $|(x) - p(x)| < \epsilon$, or equivalently, the supremum norm $\|f - p\| < \epsilon$. If f is real-valued, the polynomial function can be taken over $\mathbb{R}$.

(b)

Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and

$$\int_0^1 x^n f(x)\ dx = 0$$

for all non-negative integers n. Prove that $f = 0$.

Our goal is to show that $\int_0^1 |f|^2\ dx = 0$ we can do this by taking a sequence of polynomial approximations, p_n that converge uniformly to f so that

$$\lim_{n\to\infty} \int_0^1 p_n(x) f(x)\ dx = \int_0^1 |f(x)|^2\ dx$$

However since each p_n is a finite polynomial of the form $c_1 x^1 + \cdots + c_n x^n$ we have that

$$\lim_{n\to\infty} \int_0^1 p_n(x) f(x)\ dx = \lim_{n\to\infty} 0 = 0$$

Problem 4.3

(a)

Define strong convergence, $x_n \to x$, and weak convergencer $x_n \rightharpoonup x$, of a sequence (x_n) in a Hilbert space $\mathbb{H}$.

Strong Convergence of a sequence means that $\lim_{n\to\infty} \|x_n - x\| = 0$.

Weak convergence means that for any $y \in \mathbb{H}$, $\lim_{n\to\infty} \langle x_n, y\rangle = \langle x, y\rangle$.

(b)

If $x_n \rightharpoonup x$ weakly in $\mathbb{H}$ and $\|x_n\| \to \|x\|$, prove that $x_n \to x$ strongly.

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x\|^2 &= \lim_{n\to\infty} \langle x_n - x, x_n - x\rangle \\
&= \lim_{n\to\infty} \langle x_n, x_n\rangle - \langle x_n, x\rangle - \langle x, x_n\rangle + \langle x, x\rangle \\
&= \langle x, x\rangle - \langle x, x\rangle - \langle x, x\rangle + \langle x, x\rangle \\
&= 0
\end{aligned}$$

(c)

Give an example of a Hilbert space $\mathbb{H}$ and sequence (x_n) in $\mathbb{H}$ such that $x_n \rightharpoonup x$ weakly and

$$\|x\| < \liminf_{n\to\infty} \|x_n\|.$$

We know that the basis elements of a Hilbert space converge weakly to 0, so that

$$\|0\| < \liminf_{n\to\infty} \|e_n\| = 1$$

Problem 4.4

Suppose that $T : H \to H$ is a bounded linear operator on a complex Hilbert space H such that

$$T^* = -T, \qquad T^2 = -I$$

and $T \neq \pm iI$. Define

$$P = \frac{1}{2}(I + iT), \qquad Q = \frac{1}{2}(I - iT)$$

(a)

Prove that P, Q are orthogonal projections on H.

First lets prove they are projections,

$$\begin{aligned} P^2 &= \frac{1}{4}(I^2 - T^2 + 2iT) \\ &= \frac{1}{4}(I + I + 2iT) \\ &= \frac{1}{2}(I + iT) \\ &= P \end{aligned}$$

and

$$\begin{aligned} Q^2 &= \frac{1}{4}(I^2 - T^2 - 2iT) \\ &= \frac{1}{4}(I + I - 2iT) \\ &= \frac{1}{2}(I - iT) \\ &= Q \end{aligned}$$

Now to show they are orthogonal we will show the property $\langle Px, y\rangle = \langle x, Py\rangle$ and $\langle Qx, y\rangle = \langle x, Qy\rangle$.

$$\begin{aligned} \langle Px, y\rangle &= \langle \frac{1}{2}(I + iT)x, y\rangle \\ &= \langle \frac{1}{2}Ix, y\rangle + \langle \frac{i}{2}Tx, y\rangle \\ &= \langle x, \frac{1}{2}Iy\rangle + \langle Tx, \frac{-i}{2} - Ty\rangle \\ &= \langle x, \frac{1}{2}Iy\rangle + \langle Tx, \frac{i}{2}Ty\rangle \\ &= \langle x, \frac{1}{2}(I + iT)y\rangle \\ &= \langle x, Py\rangle \end{aligned}$$

$$\begin{aligned}
\langle Qx, y\rangle &= \langle \frac{1}{2}(I - iT)x, y\rangle \\
&= \langle \frac{1}{2}Ix, y\rangle + \langle \frac{-i}{2}Tx, y\rangle \\
&= \langle x, \frac{1}{2}Iy\rangle + \langle x, \frac{i}{2} - Ty\rangle \\
&= \langle x, \frac{1}{2}Iy\rangle + \langle x, \frac{-i}{2}Ty\rangle \\
&= \langle x, \frac{1}{2}(I - iT)y\rangle \\
&= \langle x, Qy\rangle
\end{aligned}$$

(b)

Determine the spectrum of T and classify it.

We will solve this by first finding the resolvent, the identity $T^2 = -I$ suggests the inverse of $T - \lambda I$ maybe be of the form

$$c(T + \lambda I)$$

Let's attempt to solve for the constant c,

$$\begin{aligned}
I &= (T - \lambda I)c(T + \lambda I) \\
&= c(T^2 - \lambda^2 I^2) \\
&= c(-I - \lambda^2 I)
\end{aligned}$$

so we must solve for c

$$Ic = \frac{I}{-1 - \lambda^2}$$

and so

$$c = \frac{-1}{1 + \lambda^2}$$

which is well defined for all $\lambda \neq \pm i$, however we have that $T \neq \pm iI$ so that i cannot belong to the point spectrum. Since $T^4 x = x$ we have that the range of T is onto so that the continuous and residual spectrums are empty. This means that $\pm i$ belongs to the resolvent and the spectrum for T is empty.

Problem 4.5

Let $P(\mathbb{R})$ be the Schwartz space of smooth, rapidly decreasing functions $f : \mathbb{R} \to \mathbb{C}$. Define an operator $H : P(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(\widehat{Hf})(\xi) = i\,\mathrm{sgn}(\xi)\widehat{f}(\xi) = \begin{matrix} i\hat{f}(\xi) & \text{if } \xi > 0 \\ -i\hat{f}(\xi) & \text{if } \xi < 0 \end{matrix}$$

where $\hat{f}$ denotes the Fourier transform of f.

(a)

Why is $Hf \in L^2(\mathbb{R})$ for any $f \in P(\mathbb{R})$?

Since $f \in P(\mathbb{R})$ it is in $L^2(\mathbb{R})$ and so is the Fourier transform. $P(\mathbb{R})$ so that the Fourier transform of f is in $L^2(\mathbb{R})$ so by Plancheral's theorem

$$\begin{aligned} \|Hf\|^2_{L^2(\mathbb{R})} &= \|\widehat{Hf}\|^2_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} |i\,\mathrm{sgn}(\xi)\hat{f}(\xi)|^2 \; dx \\ &= \int_0^\infty |i\hat{f}(\xi)|^2 \; dx + \int_{-\infty}^0 |-i\hat{f}(\xi)|^2 \; dx \\ &= \int_0^\infty |\hat{f}(\xi)|^2 \; dx + \int_{-\infty}^0 |\hat{f}(\xi)|^2 \; dx \\ &= \int_{-\infty}^\infty |\hat{f}(\xi)|^2 \; dx \\ &= \|\hat{f}\|^2_{L^2(\mathbb{R})} \end{aligned}$$

(b)

If $f \in P(\mathbb{R})$ and $Hf \in L^1(\mathbb{R})$, show that

$$\int_{\mathbb{R}} f(x) \; dx = 0$$

[Hint: you may want to use the Riemann-Lebesgue Lemma.]

The Riemann-Lebesgue lemma states that $\hat{f}(\xi) \to 0$ as $\xi \to \infty$ and that if $f \in L^1$ then $\hat{f} \in C$.

If $Hf \in L^1$ then we have that $\widehat{Hf} \in C$ and it decays as $|\xi| \to \infty$, since $\widehat{Hf}$ is defined in terms of $\hat{f}(\xi)$ this means that the Fourier transformation of f must go to 0 as $\xi \to \infty$.

Now since $f \in P(\mathbb{R})$ we know that $\hat{f}$ is also in $P(\mathbb{R})$ and hence continuous, since $\widehat{Hf}$ is also continuous we know that $\hat{f}(0) = 0$ so that

$$\begin{aligned}\hat{f}(0) &= 2\pi^{-n/2}\int_{\mathbb{R}} f(x)e^{-ix0}\ dx \\ &= 2\pi^{-n/2}\int_{\mathbb{R}} f(x)\ dx \\ &= 0\end{aligned}$$

and we are done!

Problem 4.6

Let Δ denote the Laplace operator in $\mathbb{R}^3$.

(a)

Prove that

$$\lim_{\epsilon \to 0} \int \frac{1}{|x} \Delta f(x)\ dx = 4\pi f(0), \quad \forall f \in P(\mathbb{R}^3).$$

(b)

Find the solution u of the Poisson problem

$$\Delta u = 4\pi f(x), \qquad \lim_{|x| \to \infty} u(x) = 0$$

for $f \in P(\mathbb{R}^3)$.

Problem 4.7

Show that the solution to the system

$$\dot{x} = 1 + x^{10}$$

goes to infinity in finite time.

Let's solve by seperation of variables

$$\int \frac{1}{1 + x^{10}} \, dx = t + c$$

Keep in mind that $\frac{1}{1+x^{10}} < \frac{1}{1+x^2}$, so that $t + c = \int \frac{1}{1+x^{10}} \, dx \leq \int \frac{1}{1+x^2} \, dx = \tan^{-1}(x)$, so that as $x \to \infty$ we have that t is always finite.

Problem 4.8

Consider the nonlinear system of ODEs:

$$\dot{x} = y - x\left((x^2+y^2)^4 - \mu\left((x^2+y^2)^2 - 1\right) - 1\right)$$
$$\dot{y} = -x - y\left((x^2+y^2)^4 - \mu\left((x^2+y^2)^2 - 1\right) - 1\right).$$

(a)

Rewrite the system in polar coordinates.

First lets rewrite $\dot{x}, \dot{y}$ in polar coordinates

$$\dot{x} = y - x(r^8 - \mu(r^4-1) - 1)$$
$$\dot{y} = -x - y(r^8 - \mu(r^4-1) - 1)$$

Using the substitution $\dot{x}x + \dot{y}y = \dot{r}r$ and $\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$

$$\begin{aligned}
\dot{r}r &= \dot{x}x + \dot{y}y \\
&= x\left(y - x(r^8 - \mu(r^4-1) - 1)\right) + y\left(-x - y(r^8 - \mu(r^4-1) - 1)\right) \\
&= xy - x^2(r^8 - \mu(r^4-1) - 1) - xy - y^2(r^8 - \mu(r^4-1) - 1) \\
&= -x^2(r^8 - \mu(r^4-1) - 1) - y^2(r^8 - \mu(r^4-1) - 1) \\
&= -(x^2+y^2)(r^8 - \mu(r^4-1) - 1) \\
&= -r^2(r^8 - \mu(r^4-1) - 1) \\
&= -r^{10} + \mu r^6 - \mu r^2 + r^2
\end{aligned}$$

and so

$$\dot{r} = -r^9 + \mu r^5 - \mu r + r$$

$$\begin{aligned}
\dot{\theta}r^2 &= x\dot{y} - y\dot{x} \\
&= x(-x - y(r^8 - \mu(r^4-1) - 1)) - y(y - x(r^8 - \mu(r^4-1) - 1)) \\
&= -x^2 - xy(r^8 - \mu(r^4-1) - 1) - y^2 + xy(r^8 - \mu(r^4-1) - 1) \\
&= -x^2 - y^2 \\
&= -r^2
\end{aligned}$$

so that

$$\dot{\theta} = -1$$

(b)

For $0 \le \mu < 1$, show that the circular region that lies within concentric circles with radius $r_{min} = 1/2$ and $r_{max} - 2$ is a trapping region. And use the Poincare-Bendixson theorem to show that there exists a stable limit cycle.

We want to find r_{min} we need a criteria for r so that $\dot{r} = -r^9 + \mu r^5 - \mu r + r > 0$,

$$0 < \mu r^5 + r$$

(c)

Show that a sub-critical Hopf Bifurcation occurs at $\mu = 1$.

September 2009

Problem 5.1

For $\epsilon > 0$, let η_ϵ denote the family of standard mollifiers on $\mathbb{R}^2$. Give $u \in L^2(\mathbb{R}^2)$, define the function

$$u_\epsilon = \eta_\epsilon * u \text{ in } \mathbb{R}^2.$$

Prove that

$$\epsilon \|Du_\epsilon\|_{L^2(\mathbb{R}^2)} \leq \|u\|_{L^2(\mathbb{R}^2)}$$

where the constant C depends on the mollifying function, but not on u.

First we establish waht Du_ϵ really is

$$\begin{aligned}
D(u * \eta_\epsilon) &= D_x \int u(x-y)\eta_\epsilon(y)\ dy \\
&= \int D_x u(x-y)\eta_\epsilon(y)\ dy \\
&= \int -D_y u(x-y)\eta_\epsilon(y)\ dy \\
&= \int u(x-y) D_y \eta_\epsilon(y)\ dy \\
&= u * D\eta_\epsilon
\end{aligned}$$

using this and Young's inequality we have

$$\begin{aligned}
\|Du_\epsilon\|_{L^2(\mathbb{R}^2)} &\leq \|u\|_{L^2(\mathbb{R}^2)} \|D\eta_\epsilon\|_{L^1(\mathbb{R}^2)} \\
&\leq \int D_x \frac{1}{\epsilon^2} \eta\left(\frac{x}{\epsilon}\right)\ dx \quad \|u\|_{L^2(\mathbb{R}^2)} \\
&\leq \int D_x \frac{1}{\epsilon^2} \eta\,(y)\,\epsilon^2\ dy \quad \|u\|_{L^2(\mathbb{R}^2)} \\
&\leq \int D_x \eta\,(y)\ \ dy \quad \|u\|_{L^2(\mathbb{R}^2)} \\
&\leq \frac{1}{\epsilon} \int D_y \eta\,(y)\ \ dy \quad \|u\|_{L^2(\mathbb{R}^2)} \\
&\leq \frac{1}{\epsilon} C \|u\|_{L^2(\mathbb{R}^2)}
\end{aligned}$$

Problem 5.2

Let $B(0,1) \subset \mathbb{R}^3$ denote the unit ball $|x| < 1$. Prove that $\log|x| \in H^1(B(0,1))$.

First we will show that $\log|x|$ is in $L^2(B(0,1))$

$$\begin{aligned}
\|\log|x|\|_{L^2(B(0,1))} &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 |\log(r)|^2 r^2 \sin(\phi)\ drd\theta d\phi \\
&= 4\pi \int_0^1 |\log(r)|^2 r^2\ dr \\
&= 4\pi \int_{-\infty}^0 |u|^2 e^{2u} e^u\ du \\
&= 4\pi \int_{-\infty}^0 |u|^2 e^{3u}\ du \\
&= -4\pi \int_{-\infty}^0 2|u|\frac{u}{|u|}\frac{e^{3u}}{3}\ du + 4\pi|u|^2\frac{e^{3u}}{3}\Big|_{-\infty}^0 \\
&= -4\pi \int_{-\infty}^0 2u\frac{e^{3u}}{3}\ du \\
&= 4\pi \int_{-\infty}^0 2\frac{e^{3u}}{9}\ du - \frac{8\pi}{9\pi} ue^{3u}\Big|_{-\infty}^0 \\
&= 4\pi \int_{-\infty}^0 2\frac{e^{3u}}{9}\ du \\
&= \frac{8\pi}{27}
\end{aligned}$$

Now we will show that $D\log(x) = \frac{1}{|x|} * \frac{x}{|x|} = \frac{x}{|x|^2}$ is in L^2.

$$\begin{aligned}
\|\frac{x}{|x|^2}\|_{L^2(B(0,1))} &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 |\frac{x}{|x|^2}|^2 r^2 \sin(\phi)\ drd\theta d\phi \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \frac{|x|^2}{|x|^4} r^2 \sin(\phi)\ drd\theta d\phi \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \frac{r^2}{r^4} r^2 \sin(\phi)\ drd\theta d\phi \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \sin(\phi)\ drd\theta d\phi \\
&< \infty
\end{aligned}$$

so we are done.

Problem 5.3

Prove that the continuous functions of compact support are a dense subspace of $L^2(\mathbb{R}^d)$.

Choose a function $f \in L^2(\mathbb{R}^2)$, since f has a finite integral we can find a bounded set A where most of the integral is located so that $\int_{\mathbb{R}^2} f < \int_A f + \epsilon$. By Urysohn's lemma we can find a continuous function u so that $u = 1$ on A and then decreases to being 0 on some set B disjoint from A. They key is we have that $\|fu - f\|_{L^2(\mathbb{R}^2)} < \epsilon$, now our goal is to show that we can find a sequence of continuous functions converging to fu. Since $fu \in L^2(A)$ for a bounded set A we can find a sequence of continuous functions $\{f_n\}$ converging to fu by utilizing the density of simple functions in $L^p(A)$ and the density of continuous functions in the set of simple functions. This gives us a sequence of continuous functions $\{f_n\}$ so that

$$\|f_n - f\|_{L^2(\mathbb{R}^2)} \leq \|f_n - fu\|_{L^2(\mathbb{R}^2)} + \|fu - f\|_{L^2(\mathbb{R}^2)}$$

Problem 5.4

There are several senses in which a sequence of bounded operators $\{T_n\}$ can converge to a bounded operator T (in a Hilbert space H).

First, there is convergence in the norm, that is, $\|T_n - T\| \to 0$, as $n \to \infty$.

Next, there is a weaker convergence, which happens to be called strong convergence, that requires $T_n f \to Tf$, as $n \to \infty$, for every vector $f \in H$.

Finally, there is weak convergence that requires $\langle T_n f, g\rangle \to \langle Tf, g\rangle$ for every pair of vectors $f, g \in H$.

(a)

Show by examples that weak convergence does not imply strong convergence, nor does strong convergence imply convergence in norm

The sequence of reaptedly applied right shift operators, R^n, on the space of $\ell^1(\mathbb{Z})$ converges weakly to 0 since

$$\langle R^n x, y\rangle \to 0$$

however $R^n x$ does converge to 0 for all x since $\|R^n x\| = \|x\|$ for all n.

The sequence of repeatedly applied left shift operators, S^n, on the space of $\ell^1(\mathbb{Z})$ converges strongly to 0, however the norm $\|S^n\| = 1$ for all n.

(b)

Show that for any bounded operator T there is a sequence $\{T_n\}$ of bounded operators of finite rank so that $T_n \to T$ strongly as $n \to \infty$.

In the case that T that H is finite dimensional the constant sequence $T_n = T$ satisfies the properties of converging strongly and being bounded along with having finite rank (the dimension of its range).

The interesting case is when the dimension of Y is infinite, for this we remember that any $f \in H$ we can decompose f by Plancheral's identity,

$$f = \sum_i \langle e_i, f\rangle e_i$$

assuming we are in a infinite dimensional space we have an infinite basis $\{e_i\}$, lets watch how T acts on a single basis element of H

$$T(e_i) = \sum_k \langle e_k, T(e_i)\rangle e_k$$

so the issue is that T can make a finite dimensional object, a single basis element, to the entire space, to get finite dimensionality for range lets define T_n where its action on a basis element is

$$T_n(e_i) = \sum_{k=0}^{n} \langle e_k, T(e_i)\rangle e_k$$

obviously this maps to a finite dimensional space, and for any element x in H we have

$$\begin{aligned} T_n(x) &= \sum_i T_n(x_i) \\ &= \sum_i \langle e_i, x \rangle T_n(e_i) \\ &= \sum_i \left(\langle \sum_{k=0}^{n} \langle e_k, T(e_i) \rangle e_k \right) \end{aligned}$$

which an infinite sum of elements in finite dimensions, so this has finite rank.

The last thing to show is that T_n converges strongly to T,

$$\begin{aligned} \lim \|T_n(e_i) - T(e_i)\|^2 &= \lim \| \sum_{k=n}^{\infty} \langle e_k, T(e_i) \rangle e_k \|^2 \\ &\leq \lim \sum_{k=n}^{\infty} \langle e_k, T(e_i) \rangle^2 \end{aligned}$$

however we know that T is bounded so that $\|T(e_i)\|^2 = \sum_{k=1}^{\infty} \langle e_k, T(e_i) \rangle^2 < M^2$ so that the sum converges, this means that as $n \to \infty, \sum_{k=n}^{\infty} T(e_i)^2$, goes to 0.

Showing strong convergence for the basis elements of H is sufficient to show strong convergence for elements in H.

Problem 5.5

Let H be a Hilbert space. Prove the following variants of the spectral theorem.

(a)

If T_1 and T_2 are two linear symmetric and compact operators on H that commute (that is $T_1T_2 = T_2T_1$), show that they

Problem 5.6

Prove that a normed linear space is complete if and only if every absolutely summable sequence is summable.

Absolutely summable means that $\sum |a_n| < \infty$.

Let's assume that our space is complete, then choose an absolutely summable sequence so that, $\sum |a_n| < \infty$, then we have that $\sum a_n is$ is a Cauchy sequence since $\|S_n - S_m\| = \| \sum_n^m a_k\| \leq \sum_n^m \|a_k\|$ which goes to 0 as n, m go to infinity since the sequence is absolutely summable.
Now lets choose a Cauchy sequence, a_n, in the space, lets extract a subsequence so that $\|a_n - a_m\| < \epsilon/2^n$. this means that $\sum \|a_n - a_{n+1}\| < \epsilon$, so that it is a absolutely summable sequence which means it must be summable, writing it out we have $\sum a_{n+1} - a_n = \lim a_n - a_1$ converges, so that a_n has a limit.

Problem 5.7

Consider the equation

$$\frac{d^2x}{dt^2} + x - \epsilon x|x| = 0.$$

(a)

Find the equation for the conserved energy.

Multiply both sides by $\dot{x}$,

$$\dot{x}\ddot{x} + \dot{x}x - \epsilon x\dot{x}|x| = 0$$

we have that the function $\ddot{x}^2/2 + x^2/2 - \epsilon|x|^3/3$ is a conserved energy.

(b)

Find the equilibrium points and the values of ϵ for which they exist.

First we write the equation as a system

$$\dot{x} = y$$
$$\dot{y} = \epsilon x|x| - x$$

clearly equilibrium points occur at $(0,0)$ and $(\pm 1/\epsilon, 0)$ for $\epsilon \neq 0$ and $(0,0)$ for $\epsilon = 0$.

(c)

There are two qualitatively different phase portraits, for different values of ϵ. CLEARLY sketch and label these phase portraits.

(d)

Show that there exist initial conditions, for any ϵ, for which solutions are periodic.

(e)

For initial data $x(0) = a, \dot{x}(0) = 0$, calculate the first two terms (in ϵa) of the Taylor expansion of the period of the orbit in the limit $\epsilon a \to 0$.

April 2010

Problem 6.1

Let (X, d) be a complete metric space, $\dot{x} \in X$, and $R > 0$. Set $D := \{x \in X \ : \ d(x, \dot{x}) \leq r\}$, and let $f : D \to X$ satisfying

$$d(f(x), f(y)) \leq kd(x, y)$$

for any $x, y \in D$, where $k \in (0, 1)$ is a constant.
Prove that if $d(\dot{x}, f(\dot{x})) \leq r(1-k)$, then f admits a unique fixed point. (Guidelines: Assume the Banach fixed point theorem, also known as the contraction mapping theorem.)

The reason we can't apply the regular contraction mapping theorem is that the construction argument used in it relies on if $x \in X$, $f(x) \in X$, where as we have our function f that takes you from D to x. Our goal is to show that f actually maps D to D, so choosing $x \in D$ we have

$$\begin{aligned} d(f(x), \dot{x}) &\leq d(f(x), f(\dot{x})) + d(f(\dot{x}), \dot{x}) \\ &\leq kd(x, \dot{x}) + r(1-k) \\ &\leq kr + r - rk \\ &\leq r \end{aligned}$$

so we see that f is actually a function from $D \to D$, allowing us to use the contraction mapping theorem

Problem 6.2

Give an example of two normed vector spaces, X and Y, and of a sequence of operators, $\{T_n\}_{n=0}^{\infty}$, $T_n \in L(X,Y)$ (where $L(X,Y)$ is the space of the continuous operators from X to Y, with the topology induced by the operator norm) such that $\{T_n\}_{n=0}^{\infty}$ is a Cauchy sequence but it does not converge in $L(X,Y)$. (Notice that Y cannot be a Banach space otherwise $L(X,Y)$ is complete.)

Let's pick the most typical incomplete normed vector spaces, the rationals, so that $X = \mathbb{Q}$ and $Y = \mathbb{Q}$.

Let's define $\rho_n(x) = (a_n)x$ where a_n is a sequence of rationals converging to $\sqrt{2}$, $\rho_n : \mathbb{Q} \to \mathbb{Q}$ this is a bounded linear operator but it converges to a bounded linear operator $\rho : \mathbb{Q} \to \mathbb{R}/\mathbb{Q}$ not contained in the space.

Problem 6.3

Let (a_n) be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} a_n^3$$

converges. Show that

$$\sum_{n=0}^{\infty} \frac{a_n}{n}$$

By Holder's inequality we have

$$\sum \frac{a_n}{n} \leq \left(\sum \left(\frac{1}{n} \right)^{3/2} \right)^{2/3} \left(\sum a_n^3 \right)^{1/3}$$

and since both terms converge we are done.

Problem 6.4

Suppose that $h : [0,1]^2 \to [0,1]^2$ is a continuously differentiable function from the square to the square with a continuously differentiable inverse h^{-1}. Define an operator T on the Hilbert space $L^2([0,1]^2)$ by the formula $T(f) = f \circ h$. Prove that T is a well-defined bounded operator on this Hilbert space.

$$\|f \circ h\|_{L^2}^2 = \int |f(h(x))| \, dx$$

Let us do a u substition for $h(x)$ so that $u = h(x)$, taking the Jacobian x with respect to u we get a well defined term that is bounded by some M on the unit box so that

$$\begin{aligned} \int |f(h(x))| \, dx &= \int |f(u)||J(u)| \, du \\ &\leq \int |f(u)| M \, du \\ &\leq M \|f\|_{L^2} \end{aligned}$$

Problem 6.5

Let $H^s(\mathbb{R})$ denote the Sobolev space of order s on the real line $\mathbb{R}$, and let

$$\|u\|_s := \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

denote the norm on $H^s(\mathbb{R})$, where $\hat{u}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} u(x) e^{-x\xi} \, dx$ denotes the Fourier transform of u.

Suppose that $r < s < t$, all real, and $\epsilon > 0$ is given. Show that there exists a constant $C > 0$ such that

$$\|u\|_s \leq \epsilon \|u\|_t + C\|u\|_r \qquad \forall \; u \in H^t(\mathbb{R})$$

Assume it is not true. Then there exists a sequence (u_n) such that

$$\|u_n\|_s > \epsilon \|u_n\|_t + n\|u_n\|_r.$$

Problem 6.6

Let $f : [0, 1] \to \mathbb{R}$. Show that f is continuous if and only if the graph of f is compact in $\mathbb{R}^2$.

Suppose that f is continuous, we will show that G the graph of f is closed and bounded.

Since f is continuous it reaches its maximum and minimum on $[0, 1]$ so that $m < f(x) < M$ and so G is contained in $[0, 1] \times [m, M]$ which is bounded. Now we must show G is closed, choose a sequence $(x_n, y_n) \to (x, y)$, $f(x_n) = y_n$, our goal is to show that (x, y) is contained in G so that $f(x) = y$. Since $\lim f(x_n) = f(\lim x_n) = f(x) = y$ by continuity of f we are done.

Now suppose the graph of f, G is compact. We will show that f is continuous, choose a sequence (x_n, y_n) so that $f(x_n) = y_n$ and $x_n \to x$. The sequence (x_n, y_n) is contained in a compact space so that there must exist a convergent subsequence converging to (x, y) and since G is closed $(x, y) \in G$, so that $f(\lim x_n) = f(x) = y = \lim y_{nk} = \lim f(x_{nk}) = \lim f(x_n)$ where the last inequality comes from f begin well defined so that if $x_n \to x, x_{nk} \to x$ then $\lim f(x_n) = \lim f(x_{nk})$.

September 2010

Problem 7.1

Let $f(x, y)$ denote a C^1 function on $\mathbb{R}^2$. Suppose that

$$f(0, 0) = 0$$

Prove that there exist two functions, $A(x, y)$ and $B(x, y)$, both continuous on $\mathbb{R}^@$ such that

$$f(x, y) = xA(x, y) + yB(x, y) \quad \forall (x, y) \in \mathbb{R}^2$$

Hint: Consider the function $g(t) = f(tx, ty)$ and express $f(x, y)$ in terms of g via the fundamental theorem of calculus.

Define $g(t) = f(tx, ty)$, this means that

$$g'(t) = x\partial_x f(tx, ty) + y\partial_y f(tx, ty)$$

Now we know that $\int_0^1 g'(t)\ dt = g(1) - g(0) = f(x, y)$ so that

$$f(x, y) = \int_0^1 g'(t)\ dt = x \int \partial_x f(tx, ty)\ dt + y \int \partial_y f(tx, ty)\ dt$$

Where both integrals are continuous because of $f \in C^1$ and integration preserving continuity.

Problem 7.2

The Fourier transform $\mathcal{F}$ of a distribution is denoted via the duality relation

$$\langle \mathcal{F}f, \phi \rangle = \langle f, \mathcal{F}^* \phi \rangle$$

for all $\phi \in C_0^\infty(\mathbb{R})$, the smooth compactly-supported test functions on $\mathbb{R}$, where

$$\mathcal{F}^*\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \phi(\xi)\ d\xi$$

Explicity compute $\mathcal{F}f$ for the function

$$\begin{aligned} f(x) = x, &\quad x > 0 \\ 0, &\quad x \leq 0 \end{aligned}$$

Since f is L^1_{loc} we can treat it as the action of a regular distribution T_f so that

$$\langle T_f, \phi \rangle = \int_{\mathbb{R}} f(x)\phi(x)\ dx = \int_0^\infty x\phi(x)\ dx$$

So lets start

$$\begin{aligned} \langle \mathcal{F}T_f, \phi \rangle &= \langle T_f, \mathcal{F}^*\phi \rangle \\ &= \int_{\mathbb{R}} f(y) \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi y} \phi(\xi)\ d\xi \right)\ dy \\ &= \int_0^\infty y \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi y} \phi(\xi)\ d\xi \right)\ dy \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_{\mathbb{R}} y e^{i\xi y} \phi(\xi)\ d\xi\ dy \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_{\mathbb{R}} \frac{-1}{i} e^{i\xi y} \phi'(\xi)\ d\xi\ dy \\ &= \frac{i}{\sqrt{2\pi}} \int_0^\infty \int_{\mathbb{R}} e^{i\xi y} \phi'(\xi)\ d\xi\ dy \\ &= i \int_0^\infty \mathcal{F}^*(\phi')\ dy \\ &= i \int_{\mathbb{R}} H(y) \mathcal{F}^*(\phi')\ dy \\ &= i \langle H, \mathcal{F}^*(\phi') \rangle \\ &= i \langle \mathcal{F}H, \phi' \rangle \\ &= -i \langle (\mathcal{F}H)', \phi \rangle \end{aligned}$$

so we see that the Fourier transform of f is $-i(\mathcal{F}H)'$.

Problem 7.3

Let $\{P_n(x)\}$ denote a sequence of polynomials on $\mathbb{R}$ such that

$$P_n \to 0 \text{ uniformly on } \mathbb{R} \text{ as } n \to \infty$$

Prove that, for n sufficiently large, all P_n are constant polynomials.

If P_n goes to 0 uniformly then we have that for $n > N$, $P_n = \sum_{i=0}^{k_n} a_{i,n} x^i \leq \epsilon$ for all x, however this is only possible if the coefficient of all the x^i terms are 0 for $i \neq 0$, since otherwise this would mean that you could choose an x large enough such that $|x^i| > \epsilon$. So this means after this N point we are only left with a sequence of the form $a_{0,n}$ which is just a sequence of constant polynomials.

Problem 7.4

For $g \in L^1(\mathbb{R}^3)$, the convolution operator G is defined on $L^2(\mathbb{R}^3)$ by

$$Gf(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} g(x-y)f(y)\ dy, \quad f \in L^2(\mathbb{R}^3)$$

Prove that the operator G with

$$g(x) = \frac{1}{4\pi}\frac{e^{-|x|}}{|x|},\ x \in \mathbb{R}^3$$

is a bounded operator on $L^2(\mathbb{R}^3)$, and the operator norm $\|G\|_{op} \leq 1$.

First let's figure out what space G maps you to, is it L^2?

$$\begin{aligned}
\|Gf(x)\|_{L^2(\mathbb{R}^3)} &= \|\frac{1}{(2\pi)^{3/2}} g \star f\|_{L^2(\mathbb{R}^3)} \\
&\leq \frac{1}{(2\pi)^{3/2}} \|g\|_{L^1(\mathbb{R}^3)} \|f\|_{L^2(\mathbb{R}^3)}
\end{aligned}$$

by Young's inequality we see that $Gf \in L^2(\mathbb{R}^3)$, we continue to find the bound on G.

$$\begin{aligned}
\|g\|_{L^1(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} \left| \frac{1}{4\pi}\frac{e^{-|x|}}{|x|} \right|\ dx \\
&= \int_{\mathbb{R}^3} \frac{1}{4\pi}\frac{e^{-|x|}}{|x|}\ dx \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{1}{4\pi}\frac{e^{-r}}{r}\ r^2 \sin(\phi)\ dr\ d\phi\ d\theta \\
&= \int_0^{\pi} \int_0^{\infty} \frac{1}{2} e^{-r} r \sin(\phi)\ dr\ d\phi \\
&= \int_0^{\infty} e^{-r} r\ dr \\
&= 1
\end{aligned}$$

Problem 7.5

Consider the map which associates to each sequence $\{x_n : n \in \mathbb{N}, x_n \in \mathbb{R}\}$ the sequence, $\{(F(\{x_n\}))_m \in \mathbb{R}\}$, defined as follows

$$\{F(\{x_n\})\}_m := \frac{x_m}{m} \quad \text{for } m = 1, 2, \dots$$

Note for $1 \leq p < \infty$, l^p denotes the space of sequences $\{x_n\}$ such that $\sum |x_n|^p < \infty$ while l^∞ denotes the space of sequences such that $\sup |x_n| < \infty$.

(a)

Determine (with proof) the values of $p \in [1, \infty]$ for which the map $F : l^p \to l^1$ is well-defined and continuous.

To find when F is well defined we need to find out what constraints we need on p to ensure that $\sum \left|\frac{x_m}{m}\right| < \infty$.

By Holder's inequality we have

$$\sum \left|\frac{x_m}{m}\right| \leq \left(\sum |x_m|^p\right)^{1/p} \left(\sum \frac{1}{m^q}\right)^{1/q}$$

The right sum only converges for $q > 1$, we know the relationship between p and q is,

$$\frac{1}{p} + \frac{1}{q} = 1$$

If q is bigger then 1 then this means that we need $\frac{1}{p}$ to be nonzero, so that $p \neq \infty$

Now what do we need for F to be continuous? Choose a sequence of sequences so that x_n, a sequence, converges to x, another sequence in the l^p norm.

We want to find what we need so that $F(x_n)$ converges to $F(x)$ with respect to the l^1 norm.

$$\begin{aligned}
\|F(x_n) - F(x)\|_{l^1} &= \sum \left|\frac{x_{n,m} - x_m}{m}\right| \\
&\leq \left(\sum |x_{n,m} - x_m|^p\right)^{1/p} \left(\sum \frac{1}{|m|^q}\right)^{1/q} \\
&\leq \|x_n - x\|_{l^p} \left(\sum \frac{1}{|m|^q}\right)^{1/q}
\end{aligned}$$

so we see yet again that we only have continuity if $p \neq \infty$.

(b)

Next, Determine the values of $q \in [1, \infty]$ for which the map $F : l^q \to l^2$ is well-defined and continuous.

Let's figure out continuity first

$$
\begin{aligned}
\|F(x_n) - F(x)\|_{l^2}^2 &= \sum \left| \frac{x_{n,m} - x_m}{m} \right|^2 \\
&= \sum \frac{|x_{n,m} - x_m|^2}{|m|^2} \\
&\leq \left(\sum |x_{n,m} - x_m|^{2p} \right)^{1/p} \left(\sum \frac{1}{|m|^{2q}} \right)^{1/q} \\
&\leq \|x_n - x\|_{l^{2p}}^2 \left(\sum \frac{1}{|m|^{2q}} \right)^{1/q}
\end{aligned}
$$

so we need that $q > 1/2$, in terms of the $\frac{1}{p} + \frac{1}{q} = 1$ relationship this means that

$$
\begin{aligned}
\frac{1}{p} &= 1 - \frac{1}{q} \\
&\geq 1 - 2 \\
&\geq -1
\end{aligned}
$$

Problem 7.6

TRUE FALSE

(a)

If (x_n) is weakly convergent then it is strongly convergent.

False, the basis elements of an infinite-dimensional Hilbert space converge weakly to 0.

(b)

If x_n is strongly convergent then it is bounded.

True, $\|x_n - x\|$ can be easily shown to be bounded

(c)

If x_n is weakly convergent then it is bounded.

True, define $\rho_n(y) = \langle x_n, y\rangle$ clearly $\|\rho_n\| = \|x_n\|$ and since for all y, $\rho_n(y)$ converges to $\langle x, y\rangle$ we have that $\rho_n(y)$ is bounded for all y. By the Uniform-Boundedness theorem this means that the collection of norms of ρ_n is bounded, which is equivalent to saying that the collection of norms of x_n are bounded as well.

(d)

If x_n is bounded then there exists a strongly convergent subsequence of x_n.

False, the basis elements of an infinite dimensional Hilbert space are bounded but there exists no strongly convergent subsequence

(e)

If x_n is bounded, there exists a weakly convergent subsequence of x_n.

True, by the Banach-Alaoglu theorem the closed unit ball is weakly compact so that any sequence has a weakly convergent subsequence

(f)

If x_n is weakly convergent and T is a bounded linear operator from H to $\mathbb{R}^d$, for some d, then $T(x_n)$ converges in $\mathbb{R}^d$.

Let us rewrite T as $T(x_n) = \sum_{i=1}^{d} T_i(x)$ where each T_i is a linear function on a Hilbert space. The Riesz representation theorem says we can express each T_i as a inner product so that, $\lim T(x_n) = \lim \sum T_i(x_n) = \lim \sum \langle y, x_n\rangle = \sum \langle y, x\rangle = T(x)$.

March 2011

Problem 8.1

Let $\omega = (0,1)$, the open unit interval in $\mathbb{R}$, and consider the sequence of functions $f_n(x) = ne^{-nx}$. Prov ethat f_n does not converge weakly to f in $L^1(\omega)$.

We can see that f_n converges strongly to 0 under the L^1 norm, so it must converge to this weakly as well, however choose $\rho = 1 \in L^\infty(\omega)$ we see that

$$\begin{aligned}\int_0^1 f_n \; dx &= \int_0^1 ne^{-nx} \; dx \\ &= -e^{-nx}\Big|_0^1 \\ &= -(e^{-n} - 1) \\ &\to 1\end{aligned}$$

so we have our contradiction.

Problem 8.2

Let $\omega = (0, 1)$, and consider the linear operator $A = -\frac{d^2}{dx^2}$ acting on the Sobolev space of functions X where

$$X = \{u = inH^2(\omega) \mid u(0) = 0, u(1) = 0\},$$

and where

$$H^2(\omega) = \Big\{ u \in L^2(\omega) = Big| \frac{du}{dx} \in L^2(\omega), \frac{d^2u}{dx^2} \in L^2(\omega) \Big\}$$

Find all the eigefunctions of A belonging to the linear span of

$$\{\cos(\alpha x), \sin(\alpha x) | \alpha \in \mathbb{R}\}$$

as well as their correponding eigenvalues.

The only functions in the span given that are eigenfunctions of A are of the form or sums of $\sin(\alpha 2\pi x)$ since these satisfy the end point restrictions and

$$A(\sin(\alpha 2\pi x) = 4\pi^2\alpha^2(\sin(\alpha 2\pi x))$$

To satisfy the end point restrictions for cos functions you need to add a constant term which disappears under differentiation.

Problem 8.3

Let $\Omega = (0,1)$, the open unit interval in $\mathbb{R}$, and set

$$v(x) = (1 + |logx|)^{-1}$$

Show that $v \in W^{1,1}(\omega)$ and that $v(0) = 0$, but that $\frac{v}{x} \notin L^1(\omega)$. (This shows the failure of Hardy's inequality in L^1.) Note that $W^{1,1}(\omega)$ blah blah blah

First since v has no blow-up at any any points besides 0 we can conclude that if $v \to 0$ as $x \to 0$ then $v \in L^1$.

$$\lim_{x\to 0} v(x) = \lim_{x\to 0} \frac{1}{1-\log(x)} = \frac{1}{1 - \lim\limits_{x\to 0}\log(x)} = \frac{1}{1+\infty} = 0$$

Now to show that $dv/dx = \frac{1}{x(1-\log(x))^2} \in L^1$

$$\begin{aligned}\int_0^1 \frac{1}{x(1-\log(x))^2}\ dx &= -\int_\infty^1 u^{-2}\ du \\ &= \int_1^\infty u^{-2}\ du \\ &\leq \infty\end{aligned}$$

so $v \in W^{1,1}$. However $v(x)/x \notin W^{1,1}$ since

$$\begin{aligned}\int_0^1 \frac{v(x)}{x}\ dx &= \int_1^\infty u^{-1}\ du \\ &= \infty\end{aligned}$$

Problem 8.4

Let $f(x)$ be a periodic continuous function on $\mathbb{R}$ with period 2π. Show that

$$\hat{f}(\xi) = \sum b_n \tau_n \delta \text{ in } \mathcal{D}'$$

in the sense of distributions. Relate b_n to the coefficients of the Fourier Series. $\tau_n\delta(x) = \delta(x+n)$.

Our goal is to show that

$$\langle \mathcal{F}f, \phi\rangle = \langle \sum b_n \tau_n \delta, \phi\rangle \quad \forall \phi \in C_0^\infty(\mathbb{R})$$

lets start with the left side

$$\langle \mathcal{F}f, \phi\rangle = \langle f, \mathcal{F}^*\phi\rangle$$

$$= \int_{-\infty}^{\infty} f(x)\mathcal{F}^*\phi(x)\, dx$$

by uniform convergence of the Fourier transform of a continuous function

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_n e^{inx}\mathcal{F}^*\phi(x)\, dx$$

by Lebesgue dominated convergence theorem

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} c_n e^{inx}\mathcal{F}^*\phi(x)\, dx$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} c_n e^{inx} \int_{-\infty}^{\infty} e^{i\xi x}\phi(\xi) d\xi\, dx$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_n e^{inx} e^{i\xi x}\phi(\xi) d\xi\, dx$$

$$= \sum_{n=-\infty}^{\infty} \lim_{R\to\infty} \int_{-R}^{R} \int_{-\infty}^{\infty} c_n e^{inx} e^{i\xi x}\phi(\xi) d\xi\, dx$$

by Lebesgue dominated convergence theorem

$$= \sum_{n=-\infty}^{\infty} \lim_{R\to\infty} \int_{-\infty}^{\infty} \int_{-R}^{R} c_n e^{inx} e^{i\xi x}\phi(\xi)\, dx\, d\xi$$

$$= \sum_{n=-\infty}^{\infty} \lim_{R\to\infty} \int_{-\infty}^{\infty} c_n \int_{-R}^{R} e^{inx} e^{i\xi x}\, dx \phi(\xi)\, d\xi$$

$$= \sum_{n=-\infty}^{\infty} \lim_{R\to\infty} \int_{-\infty}^{\infty} \int_{-R}^{R} e^{i(n+\xi)x}\, dx c_n \phi(\xi)\, d\xi$$

$$= \sum_{n=-\infty}^{\infty} \lim_{R\to\infty} \left\langle c_n \int_{-R}^{R} e^{i(n+\xi)x} dx, \phi(\xi) \right\rangle$$

definition of delta distribution

$$= \sum_{n=-\infty}^{\infty} \langle c_n \delta_n, \phi(\xi)\rangle$$

Problem 8.5

$f(x)$ periodic continuous functions on $\mathbb{R}$ with peroid 2π. Prove there is a finite Fourier series

$$\phi(x) = a_o + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

such that

$$|\phi(x) - f(x)| < \epsilon$$

Problem 8.6

Show that the closed unit ball of $\{u \in C^{0,\alpha}\}$ is a compact set in $C([0,1])$.

First we will show that u is uniformly bounded

$$|u(x)| \leq \|u\|_{C^{0,\alpha}} \leq 1$$

similarly u is equicontinuous

$$\frac{|u(x) - u(y)|}{(x-y)^\alpha} \leq \|u\|_{C^{0,\alpha}} \leq 1$$

so that

$$|u(x) - u(y)| \leq (x-y)^\alpha$$

and so by Arzelá-Ascoli we are done.

September 2011

Problem 9.1

Let (X, d) be a metric space and let (x_n) be a sequence in X. For the purpose of this problem adopt the following definition: $x \in X$ is called a *cluster point* of (x_n) iff there exists a subsequence $(x_{n_k})_{k\geq 0}$ such that $\lim_k x_{n_k} = x$.

(a)

Let $(a_n)_{n\geq 0}$ be a sequence of distinct points in X. Construct a sequence in $(x_n)_{n\geq 0}$ in X such that for all $k = 0, 1, 2, \ldots,$ a_k is a cluster point of (x_n).

$$(x_n) = a_1, a_1, a_2, a_1, a_2, a_3, a_1, a_2, a_3, a_4, \ldots$$

(b)

Can a sequence (x_n) in a metric space have an *uncountable* number of cluster points? Prove your answer. (If you answer yes, give an example with proof. If you answer no, prove that such a sequence cannot exist.) You may use without proof that $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable.

Yes. The set $[0, 1] \subset \mathbb{R}$ is uncountable, and the set $[0, 1] \subset \mathbb{Q}$ is countable and is dense in $[0, 1] \subset \mathbb{R}$. We can list the elements of $[0, 1] \subset \mathbb{Q}$ as a sequence:

$$(x_n) = 0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \ldots$$

Every $y \in [0, 1] \subset \mathbb{R}$ is a cluster point of this sequence, so it has an uncountable number of cluster points.

Problem 9.2

Let X be a real Banach space and X^* its Banach space dual. For any bounded linear operator $T \in \mathcal{B}(X)$, and $\phi \in X^*$, define the functional $T^*\phi$ by

$$T^*\phi(x) = \phi(Tx), \qquad \text{for all } x \in X.$$

(a)

Prove that T^* is a bounded operator on X^* with $\|T^*\| \leq \|T\|$.

$$\begin{aligned} T &: X \to X \\ T^* &: X^* \to X^* \\ \phi &: X \to \mathbb{R} \\ T^*\phi &: X \to \mathbb{R} \end{aligned}$$

$$\begin{aligned} \|T^*\| &= \sup_{\|\phi\|=1} \|T^*\phi\| \\ &= \sup_{\|\phi\|=1} \sup_{\|x\|=1} |T^*\phi(x)| \\ &= \sup_{\|\phi\|=1} \sup_{\|x\|=1} |\phi(Tx)| \\ &\leq \sup_{\|\phi\|=1} \sup_{\|x\|=1} \|\phi\|\|Tx\| \\ &\leq \sup_{\|x\|=1} \|Tx\| \\ &\leq \|T\| \end{aligned}$$

(b)

Suppose $0 \neq \lambda \in \mathbb{R}$ is an eigenvalue of T. Prove that λ is also an eigenvalue of T^*. (**Hint 1:** first prove the result for $\lambda = 1$. **Hint 2:** For $\phi \in X^*$, consider the sequence of Cesàro means $\psi_N = N^{-1}\sum_{n=1}^{N} \phi_n$, of the sequence ϕ_n defined by $\phi_n(x) = \phi(T^n x)$.)

Problem 9.3

Let $\mathcal{H}$ be a complex Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear transformations (operators) on $\mathcal{H}$ considered with the operator norm.

(a)

What does it mean for $A \in \mathcal{B}(\mathcal{H})$ to be *compact*? Give a definition of compactness of an operator A in terms of properties of the image on bounded sets, e.g. the set $\{Ax \mid x \in \mathcal{H},\ \|x\| \le 1\}$.

A maps bounded sets to precompact sets.

(b)

Suppose $\mathcal{H}$ is separable and let $\{e_n\}_{n\ge 0}$ be an orthonormal basis of $\mathcal{H}$. For $n \ge 0$, let P_n denote the orthogonal projection onto the subspace spanned by $e_0, \dots, e_n$. Prove that $A \in \mathcal{B}(\mathcal{H})$ is compact iff the sequence $(P_n A)_{n\ge 0}$ converges to A in norm.

$(P_n A)$ converges to A in norm $\Rightarrow$ A is compact

Choose a bounded subsequence (x_n), $\|x_n\| \le M$. Consider $P_1 A$. This is a finite rank operator, so it is compact. Therefore, we have a convergent subsequence $(x_{n,1})$. Do the same for all n, then use a diagonal argument and create a subsequence $(x_{k,k})$. We want to show $Ax_{k,k}$ converges. To do this, we will show that it is Cauchy.

$$\begin{aligned}\|Ax_{k,k} - Ax_{j,j}\| &\le \|Ax_{k,k} - P_n Ax_{k,k}\| + \|P_n Ax_{k,k} - P_n Ax_{j,j}\| + \|P_n Ax_{j,j} - Ax_{j,j}\| \\ &\le \|A - P_n A\|M + \epsilon + \|P_n A - A\|M \to 0.\end{aligned}$$

A compact $\Rightarrow$ $(P_n A)$ converges to A in norm

Let $E = \{x \mid \|x\| = 1\}$. E is bounded, so $A(E)$ is precompact. Also, there exists a sequence $(x_n) \subset E$ such that

$$\lim_{n\to\infty} \|(P_n A - A)x_n\| = \sup_{\|x\|=1} \|(P_n A - A)x\| = \|P_n A - A\|.$$

Then

$$\|(P_n A - A)x\| = \left\| \sum_{k=n+1}^{\infty} \langle e_k, Ax\rangle\, e_k \right\| \to 0.$$

This is because of Theorem 9.17, which says that if $A(E)$ is precompact, then for every orthonormal set $\{e_n \mid n \in \mathbb{N}\}$ and every $\epsilon > 0$, there is an N such that

$$\sum_{n=N+1}^{\infty} |\langle e_n, x\rangle|^2 < \epsilon \qquad \text{for all } x \in A(E).$$

Problem 9.4

Let $\Omega \in \mathbb{R}^n$ be open, bounded, and smooth. Suppose that $\{f_j\}_{j=1}^{\infty} \subset L^2(\Omega)$ and $f_j \rightharpoonup g_1$ weakly in $L^2(\Omega)$ and that $f_j(x) \to g_2(x)$ a.e. in Ω. Show that $g_1 = g_2$ a.e. (**Hint:** Use Egoroff's theorem which states that given our assumptions, for all $\epsilon > 0$, there exists $E \subset \Omega$ such that $\lambda(E) < \epsilon$ and $f_j \to g_2$ uniformly on E^c.)

(See Theorem 1.46 in Shkoller's 201C Notes.)

$\Omega \subset \mathbb{R}^n$ smooth, bounded, open. $f_j \rightharpoonup g_1 \in L^2(\Omega)$, $f_j(x) \to g_2(x)$ a.e. This gives us that f_j converges uniformly to g_2 on the set E^c where the measure of E is less than ϵ. We start out with $g_1(x)$:

$$\begin{aligned} g_1(x) &\overset{\text{a.e.}}{=} \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{B_\epsilon(x)} g_1(y)\ dy && \text{(LDT)} \\ &= \lim_{\epsilon\to 0} \lim_{j\to\infty} \frac{1}{\epsilon} \int_{B_\epsilon(x)} f_j(y)\ dy && \text{weak convergence} \\ &= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{B_\epsilon(x)} g_2(y)\ dy && \text{Egoroff's Theorem} \\ &= g_2(x). \end{aligned}$$

So here's the explaination for every step. To utilize the Lebesgue Diffrentiation Theorem all you need is for your function to be locally summable, then the first equivalence I wrote down works a.e. The second equivalence comes from weak convergence. The third equivalence comes from uniform convergence of f_j to g_2 at any point that isn't a singularity, which im thinking must be a.e. since if it wasnt that means g_2 would have singularities everywhere and it would be impossible for f_j, an L^2 function, to converge uniformly to it on any set since uniform convergence implies L^2 convergence so that on the set E^c you need g_2 to be L^2. Anyways, I think all you need is to assume that there aren't singularities a.e., which I think we'll end up justifying by using that on the set E^c, g_2 has to be a L^2 function.

For the fourth equivalence all we need to justify it is that g_2 is locally summable around x, which it is because since x isnt a singularity. We can include a neighborhood around it in the set E so that g_2 is a uniform limit of L^2 functions f_j so that it is also in L^2, and since E is a bounded domain g_2 is in L^1.

Problem 9.5

Let $u(x) = (1 + |\log x|)^{-1}$. Prove that $u \in W^{1,1}(0,1)$, $u(0) = 0$, but $\frac{u}{x} \notin L^1(0,1)$.

(Same as March 2011, Problem 3)

First since u has no blow-up at any any points besides 0 we can conclude that if $u \to 0$ as $x \to 0$ then $u \in L^1$.

$$\lim_{x\to 0} u(x) = \lim_{x\to 0} \frac{1}{1-\log(x)} = \frac{1}{1 - \lim\limits_{x\to 0}\log(x)} = \frac{1}{1+\infty} = 0$$

Now to show that $du/dx = \frac{1}{x(1-\log(x))^2} \in L^1$

$$\begin{aligned}\int_0^1 \frac{1}{x(1-\log(x))^2}\ dx &= -\int_\infty^1 v^{-2}\ dv \\ &= \int_1^\infty v^{-2}\ du \\ &\leq \infty\end{aligned}$$

so $u \in W^{1,1}$. However $u(x)/x \notin W^{1,1}$ since

$$\begin{aligned}\int_0^1 \frac{u(x)}{x}\ dx &= \int_1^\infty v^{-1}\ dv \\ &= \infty\end{aligned}$$

Problem 9.6

Let $H = \left\{ f \in L^2(0, 2\pi) \mid \int_0^{2\pi} f(x)\,dx = 0 \right\}$. We define the operator Λ as follows:

$$(\Lambda f)(x) = \int_0^x f(y)\,dy.$$

(a)

Prove that $\Lambda : H \to L^2(0, 2\pi)$ is continuous.

We want to show

$$\|\Lambda f - \Lambda g\|_{L^2}^2 \le C\|f - g\|_{L^2}^2.$$

$$\begin{aligned}
\int_0^{2\pi} (\Lambda f - \Lambda g)^2\,dx &= \int_0^{2\pi} \int_0^x [f(y) - g(y)]^2\,dy\,dx \\
&\le \int_0^{2\pi} \int_0^{2\pi} |f(y) - g(y)|^2\,dy\,dx \\
&\le 2\pi\|f - g\|_{L^2}^2
\end{aligned}$$

(b)

Use the Fourier series to show that the following estimate holds:

$$\|\Lambda f\|_{H_0^1(0,2\pi)} \le C\|f\|_{L^2(0,2\pi)},$$

where C denotes a constant which depends only on the domain $(0, 2\pi)$. (Recall that $\|u\|_{H_0^1(0,2\pi)}^2 = \int_0^{2\pi} \left|\frac{du}{dx}(x)\right|^2\,dx$.)

Cheater way:

$$\begin{aligned}
\int_0^{2\pi} \left|\frac{d}{dx}\Lambda f\right|^2\,dx &= \int_0^{2\pi} \left|\frac{d}{dx}\int_0^x f(y)\,dy\right|^2\,dx \\
&= \int_0^{2\pi} |f(x)|^2\,dx = \|f\|_{L^2}^2
\end{aligned}$$

Fourier series way:

$$\|D\Lambda f\|_{L^2}^2 = \|\mathcal{F}(D\Lambda f)\|_{L^2}^2$$

Goal: find $\widehat{\Lambda f}(n)$.

$$\begin{aligned}
\widehat{\Lambda f}(n) &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \Lambda f(x) e^{-inx}\,dx \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \int_0^x f(y)\,dy e^{-inx}\,dx \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \int_0^x f(y) e^{-inx}\,dy\,dx \\
&= \frac{1}{\sqrt{2\pi}} \int_y^{2\pi} \int_0^x f(y) e^{-inx}\,dx\,dy \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(y) \cdot \left.\frac{e^{-inx}}{-in}\right|_y^{2\pi} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(y) \left(\frac{1}{-in} + \frac{e^{-iny}}{in} \right) dy \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(y) \frac{e^{-iny}}{in}\,dy \\
&= \frac{\hat{f}_n}{in}
\end{aligned}$$

$$\begin{aligned}
\|D\Lambda f\|_{L^2}^2 &= \|\mathcal{F}(D\Lambda f)\|_{L^2}^2 \\
&= \sum |\widehat{D\Lambda f}(n)|^2 \\
&= \sum |in\Lambda f(n)|^2 \\
&= \sum \left| in \frac{\hat{f}(n)}{in} \right|^2 \\
&= \sum |\hat{f}(n)|^2 \\
&= \|f\|_{L^2}^2
\end{aligned}$$

Problem 9.7

Consider the system

$$\dot{x} = \mu x + y + \tan x, \qquad \dot{y} = x - y.$$

(a)

Show that a bifurcation occurs at the origin $(x, y) = (0, 0)$, and determine the critical value $\mu = \mu_c$ at which the bifurcation occurs.

(b)

Determine the type of bifurcation that occurs at $\mu = \mu_c$. Do this (i) analytically and (ii) graphically (sketch the appropriate phase portraits for μ slightly less than; equal to; and slightly greater than μ_c).

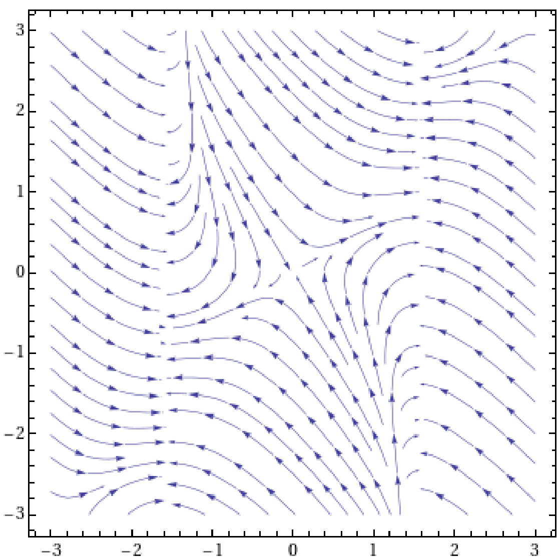

Figure 1: $\dot{x} = -1 \cdot x + y + \tan x,\ \dot{y} = x - y$

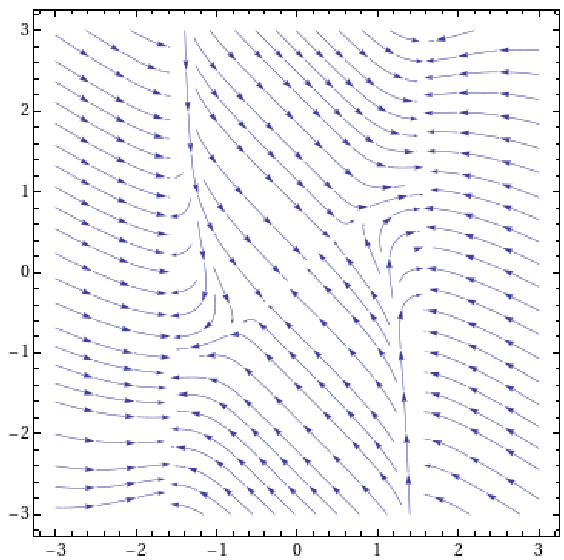

Figure 2: $\dot{x} = -2 \cdot x + y + \tan x,\ \dot{y} = x - y$

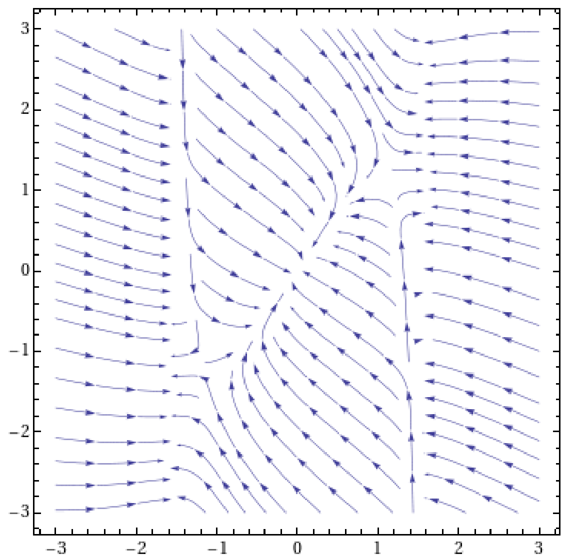

Figure 3: $\dot{x} = -3 \cdot x + y + \tan x,\ \dot{y} = x - y$

Problem 9.8

Consider the differential equation

$$\ddot{x} + x - x^3 = 0,$$

with the initial condition $x(0) = \epsilon$, $\dot{x}(0) = 0$, where $\epsilon \ll 1$. Use "two-timing" and perturbation theory to approximate the frequency of oscillation to order ϵ^2.

(a)

Make a change of variables so that the differential equation is in the form $\ddot{z} + z + \epsilon h(z, \dot{z}) = 0$, i.e. in a form where ϵ appears naturally in the equation as a perturbation parameter.

Let $z = x\epsilon^c$.

$$\begin{aligned}
x &= \epsilon^{-c} z \\
\ddot{x} &= \epsilon^{-c} \ddot{z} \\
x^3 &= \epsilon^{-3c} z^3 \\
\epsilon^{-c}\ddot{z} + \epsilon^{-c} z - \epsilon^{-3c} z^3 &= 0 \\
\ddot{z} + z - \epsilon^{-2c} z^3 &= 0 \\
-2c &= 1 \\
c &= -\frac{1}{2} \\
z &= \frac{x}{\sqrt{\epsilon}}
\end{aligned}$$

(b)

Rewrite the equation assuming two time scales, a fast time $\tau = t$ and a slow one $T = \epsilon\tau$, and the solution form $z(t, \epsilon) = z_0(\tau, T) + \epsilon z_1(\tau, T) + O(\epsilon^2)$.

(c)

Show that the order 0 (i.e., $O(1)$) solution takes the form

$$z_0(\tau, T) = r(T)\cos(\tau + \phi(T)).$$

(d)

Use the order 1 (i.e., $O(\epsilon)$) equation to determine the frequency of oscillation to order ϵ^2. (**Hint:** The order 1 (i.e., $O(\epsilon)$) equation contains resonant terms, which would cause the solution to grow without bound as $t \to \infty$. A solution that remains bounded for large τ is obtained by setting the coefficients of the resonant terms to zero. This yields equations that can be used to find the order ϵ^2 correction for the frequency of oscillation. Note: Be sure to look for "hidden" resonance terms. It may be helpful to use the trig identity $\cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta)$.)