

UC Davis
Applied Math
Prelim Solutions
"Back of the Napkin" style

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If you find a mistake,
please send the correct
solution to the author.

F'013 #1

$$h'' + k^2 h = 0$$

$$k = k(x) = k_1 + (k_2 - k_1) \tanh\left(\frac{x}{L}\right) \quad \alpha < k_1 < k_2$$

$$X := \frac{x}{L} \quad \kappa(X) := k(x) \quad H(X) := h(x)$$

$$\frac{1}{L^2} H'' + \kappa^2 H = 0$$

$$H' = \exp(Lu_0 + u_1)$$

$$\frac{1}{L^2} \left[(Lu_0'' + u_1'') + (Lu_0' + u_1')^2 \right] + \kappa^2 = 0$$

Assume
 κ is $O(1)$

$$O(1): \quad (u_0')^2 + \kappa^2 = 0$$

$$(eikonal) \quad u_0' = \pm i\kappa$$

$$u_0 = C \pm i \int \kappa(t) dt$$

$$O(\epsilon): \quad u_0'' + 2u_0' u_1' = 0$$

$$(transport) \quad \pm i\kappa' \pm 2i\kappa u_1' = 0$$

$$\kappa' + 2\kappa u_1' = 0$$

$$u_1' = -\frac{1}{2} \frac{\kappa'}{\kappa}$$

$$u_1 = D - \frac{1}{2} \ln \kappa$$

$$H = \exp\left(CL \pm iL \int \kappa(t) dt + D - \frac{1}{2} \ln \kappa \right)$$

$$= \frac{1}{\sqrt{\kappa}} \exp\left(CL + D \pm iL \int \kappa(t) dt \right)$$

$$h(x) = \frac{1}{\sqrt{k(x)}} \exp\left(CL + D \pm i \int k(t) dt \right)$$

$$= \frac{1}{\sqrt{k(x)}} \exp(CL + D) \exp\left(\pm i \int \kappa(t) dt \right)$$

In part (c), we are given info to conclude that $(k_2 - k_1) \approx 0$ } odd error in question?
 $\therefore k(t) \approx k_1$

$$A^2 = \lim_{x \rightarrow -\infty} |h(x)|^2 = \frac{1}{k_1} \exp(2CL + 2D)$$

$$\therefore h(x) = A \exp(ik_1 x)$$

F'013 #2

$$\epsilon y'' + \sqrt{x} y' + y = 0$$

$$y(0) = 0, y(1) = 1$$

INNER SOLUTION

$$\text{Let } x = \epsilon^2 X, Y(X) := y(x)$$

$$\epsilon^{1-2\alpha} Y'' + \epsilon^{-\frac{1}{2}\alpha} \sqrt{X} Y' + Y = 0, Y(0) = 0$$

$$\alpha = \frac{2}{3}$$

$$\epsilon^{-\frac{1}{3}} Y'' + \epsilon^{-\frac{1}{3}} \sqrt{X} Y' + Y = 0$$

$$Y'' + \sqrt{X} Y' + \epsilon^{\frac{1}{3}} Y = 0$$

$$O(1): Y'' + \sqrt{X} Y' = 0$$

$$\frac{Y''}{Y'} = -\sqrt{X}$$

$$\ln Y' = -\frac{2}{3} X^{\frac{3}{2}} + C$$

$$Y' = c e^{-\frac{2}{3} X^{\frac{3}{2}}}$$

$$Y = c \int_0^X e^{-\frac{2}{3} t^{\frac{3}{2}}} dt + d$$

$$Y(0) = 0 \Rightarrow d = 0$$

$$\therefore Y = c \int_0^X e^{-\frac{2}{3} t^{\frac{3}{2}}} dt$$

$$\begin{aligned} 1-2\alpha &= -\frac{1}{2}\alpha \\ 1 &= \frac{3}{2}\alpha \\ \frac{2}{3} &= \alpha \end{aligned}$$

OUTER SOLUTION

$$\sqrt{x} y' + y = 0 \quad y(1) = 1$$

$$\frac{y'}{y} = -\frac{1}{\sqrt{x}}$$

$$\ln y = -2\sqrt{x} + a$$

$$y = a e^{-2\sqrt{x}}$$

$$1 = y(1) = a e^{-2} \Rightarrow a = e^2$$

$$\therefore y(x) = e^{2-2\sqrt{x}}$$

MATCH

$$\lim_{X \rightarrow \infty} Y(X) = \lim_{x \rightarrow 0^+} y(x)$$

$$c \int_0^{\infty} e^{-\frac{2}{3} t^{\frac{3}{2}}} dt = e^2$$

$$c = \frac{e^2}{\int_0^{\infty} e^{-\frac{2}{3} t^{\frac{3}{2}}} dt}$$

SOLUTION

$$y(x) + Y\left(\frac{x}{\epsilon^2}\right) - e^2$$

$$e^{2-2\sqrt{x}} + \frac{e^2}{\int_0^{\frac{x}{\epsilon^2}} e^{-\frac{2}{3} t^{\frac{3}{2}}} dt} - e^2$$

F'013 #3

$$u_{tt} = a^2(u_{rr} + \frac{1}{r}u_r)$$

$$u(R, t) = 0$$

$$u(r, t) := f(r)g(t)$$

$$fg'' = a^2f''g + \frac{1}{r}f'g$$

$$\frac{g''}{g} = a^2 \frac{f''}{f} + \frac{1}{r} \frac{f'}{f} = -n^2 \quad n > 0$$

$$a^2 f'' + \frac{1}{r} f' = -n^2 f$$

$$g'' + n^2 g = 0$$

$$a^2 f'' + \frac{1}{r} f' + n^2 f = 0$$

not sure how to proceed other than to
try to solve these ODEs

F'013 #4 (Part 1)

$$f'' + \lambda f = 0$$

$$f'(0) = f(l) = 0$$

Claim: $\lambda > 0$

PF/

$$0 < \frac{\int_0^l (f')^2 dx - \int_0^l f f'' dx + [f f']_0^l}{\int_0^l f^2 dx} = \frac{-\int_0^l f(-\lambda f) dx}{\int_0^l f^2 dx} = \frac{\lambda \int_0^l f^2 dx}{\int_0^l f^2 dx} = \lambda //$$

wlog let $\lambda = \frac{\pi^2 n^2}{4l^2}$, $n \in \mathbb{R}$.

$$f'' + \frac{\pi^2 n^2}{4l^2} f = 0$$

solved by $f(x) = a \cos \frac{\pi n}{2l} x + b \sin \frac{\pi n}{2l} x$

$$0 = f(l) = a \cos \frac{\pi n}{2} + b \sin \frac{\pi n}{2} = a \cos \frac{\pi n}{2} \Rightarrow a = 0 \text{ or } n \in 2\mathbb{Z} - 1$$

$$0 = f'(0) = -\frac{\pi n a}{2l} \sin \frac{\pi n}{2l} \cdot 0 + \frac{\pi n b}{2l} \cos \frac{\pi n}{2l} \cdot 0 = \frac{\pi n b}{2l} \Rightarrow b = 0.$$

$$f(x) = 0 \text{ or } f(x) = a \cos \frac{\pi(2k-1)}{2l} x, \quad k \in \mathbb{N}$$

$$\text{Normalize: } a^2 \int_0^l \cos^2 \frac{\pi(2k-1)}{2l} x dx = \frac{l}{2} a^2 = 1 \Rightarrow a = \sqrt{\frac{2}{l}}$$

eigenvalues: $\lambda_k = \frac{\pi^2 (2k-1)^2}{4l^2}, k \in \mathbb{N}$

eigenfunctions: $\sqrt{\frac{2}{l}} \cos \frac{\pi(2k-1)}{2l} x$

F'013 #4 (Part 2)

approximate $l^2 - x^2$ with
 $\{\phi_k\} = \left\{ \sqrt{\frac{2}{l}} \cos \frac{\pi(2k-1)x}{2l} \right\}$

project: $g \mapsto \sum_{k=1}^{\infty} \langle \phi_k, g \rangle \phi_k$. Let $\kappa := \frac{\pi(2k-1)}{2l}$

$$\begin{aligned} \langle \phi_k, g \rangle &= \sqrt{\frac{2}{l}} \int_0^l (l^2 - x^2) \cos \kappa x \, dx \\ &= \sqrt{\frac{2}{l}} \left[-\frac{x^2}{\kappa} \sin \kappa x - \frac{2x}{\kappa^2} \cos \kappa x + \frac{2}{\kappa^3} \sin \kappa x \right]_0^l \\ &= \sqrt{\frac{2}{l}} \left[-\frac{x^2}{\kappa} (-1)^{k+1} + \frac{2x}{\kappa^3} (-1)^{k+1} + \frac{2x}{\kappa^2} \right] \end{aligned}$$

(put these pieces together for answer)

$$\begin{aligned}\dot{x} &= ax + y - x f(x^2 + y^2) \\ \dot{y} &= -x + ay - y f(x^2 + y^2)\end{aligned}$$

$$a \in \mathbb{R}, f \in C^0, f(0) = 0, f(u) \geq u^{1/2}$$

$$\begin{aligned}2x\dot{x} + 2y\dot{y} &= 2ax^2 + 2xy - x^2 f(x^2 + y^2) - 2xy + 2ay^2 - y^2 f(x^2 + y^2) \\ &= 2a(x^2 + y^2) - (x^2 + y^2) f(x^2 + y^2) = (x^2 + y^2) [2a - f(x^2 + y^2)]\end{aligned}$$

fixed points:

$$2x(0) + 2y(0) = (x^2 + y^2) [2a - f(x^2 + y^2)]$$

$$x^2 + y^2 = 0 \quad \text{or} \quad f(x^2 + y^2) = 2a$$

$$(0, 0)$$

$$\text{would imply } \begin{cases} 0 = \dot{x} = ax + y - 2ax = y - ax \\ 0 = \dot{y} = -x + ay - 2ay = -x - ay \end{cases}$$

$$\begin{cases} 0 = \dot{x} = ax + y - 2ax = y - ax \\ 0 = \dot{y} = -x + ay - 2ay = -x - ay \end{cases}$$

$$\therefore y = ax \text{ and } x = -ay$$

$$\therefore y = -a^2 y$$

$$\therefore y = 0, x = 0$$

\therefore only fixed pt: $(0, 0)$

Linear stability: assume $f \in C^1$

$$J(x, y) = \begin{bmatrix} a - f(x^2 + y^2) - 2x^2 f'(x^2 + y^2) & 1 - 2xy f'(x^2 + y^2) \\ -1 - 2xy f'(x^2 + y^2) & a - f(x^2 + y^2) - 2y^2 f'(x^2 + y^2) \end{bmatrix}$$

$$J(0, 0) = \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix}$$

$$\lambda = a \pm i$$

stable spiral for $a < 0$

unstable spiral for $a > 0$

F '013 #5 Page 2

We have that $(0,0)$ is the only fixed pt and it is a spiral source for $a > 0$.
 Now recall $2x\dot{x} + 2y\dot{y} = (x^2 + y^2)[2a - f(x^2 + y^2)]$

$$\Rightarrow \frac{d}{dt}[x^2 + y^2] = (x^2 + y^2)[2a - f(x^2 + y^2)]$$

Consider the ^{closed} region bdd by $x^2 + y^2 = 9a^2$ with the neighborhood of $(0,0)$
 On the outer bdy of this region,

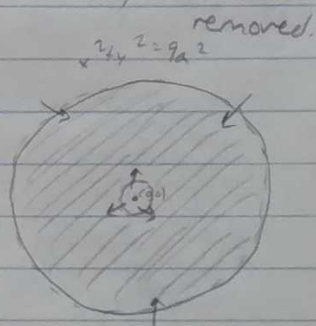
$$\frac{d}{dt}[x^2 + y^2] = (x^2 + y^2)[2a - f(x^2 + y^2)]$$

$$= 9a^2[2a - f(9a^2)]$$

$$\leq 9a^2[2a - 3a]$$

$$= -9a^3$$

< 0 . \therefore all trajectories flow inward



On the inner bdy of this region, all trajectories flow inward (spiral source at $(0,0)$)

There are no fixed pts inside the region.

\therefore By P-B a limit cycle must exist.

Special case: $f(u) = u^2$.

$$0 = \frac{d}{dt}[x^2 + y^2] = (x^2 + y^2)[2a - f(x^2 + y^2)]$$

$$0 = x^2 + y^2 \quad \text{or} \quad 0 = 2a - f(x^2 + y^2)$$

not in region

$$2a = f(x^2 + y^2)$$

$$2a = \sqrt{x^2 + y^2}$$

$$\boxed{4a^2 = x^2 + y^2}$$

F'013 #6

$$\frac{d^2 r}{d\theta^2} + r = \frac{1}{L} + \epsilon L r^2$$

$$\left\{ \begin{array}{l} \frac{dr}{d\theta} := s \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{ds}{d\theta} = -r + \frac{1}{L} + \epsilon L r^2 \end{array} \right.$$

$$\text{fixed pts: } \left\{ \begin{array}{l} 0 = s \\ 0 = -r + \frac{1}{L} + \epsilon L r^2 \end{array} \right.$$

$$\epsilon L r^2 - r + \frac{1}{L} = 0$$

$$r = \frac{1 \pm \sqrt{1 - 4\epsilon}}{2\epsilon L}$$

$$J(r, s) = \begin{bmatrix} 0 & 1 \\ -1 + 2\epsilon L r & 0 \end{bmatrix}$$

$$J\left(\frac{1 \pm \sqrt{1 - 4\epsilon}}{2\epsilon L}, 0\right) = \begin{bmatrix} 0 & 1 \\ \pm \sqrt{1 - 4\epsilon} & 0 \end{bmatrix}$$

saddle at $\left(\frac{1 + \sqrt{1 - 4\epsilon}}{2\epsilon L}, 0\right)$ as $\epsilon \rightarrow 0^+$: $(\infty, 0)$

center at $\left(\frac{1 - \sqrt{1 - 4\epsilon}}{2\epsilon L}, 0\right)$ as $\epsilon \rightarrow 0^+$: $\left(\frac{1}{L}, 0\right)$
(conservative system)

