UC Davis
Applied Math
Prelim Solutions
"Back of the Napkin" style

Compiled 9/13/2016
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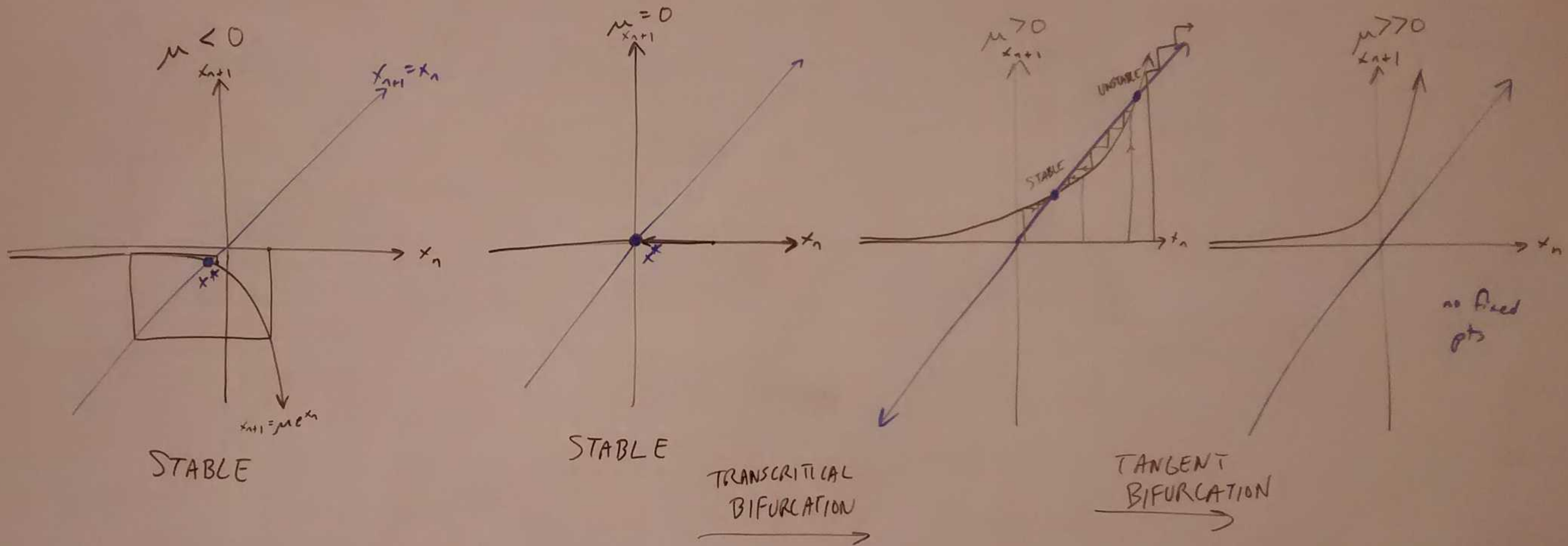
If you find a mistake,
please send the correct
solution to the author.

F'012 #1

$$x_{n+1} = \mu e^{x_n} \qquad \mu \in \mathbb{R}, \quad n \in \mathbb{N} \cup \{0\}$$



Stok
circula
Laplac
$-\frac{\partial V}{\partial x} =$
Lyapunov
Poincaré
Stirlin
$L -$
Practice Pro

F'012 #2

$$\dot{x}_1 = x_2$$

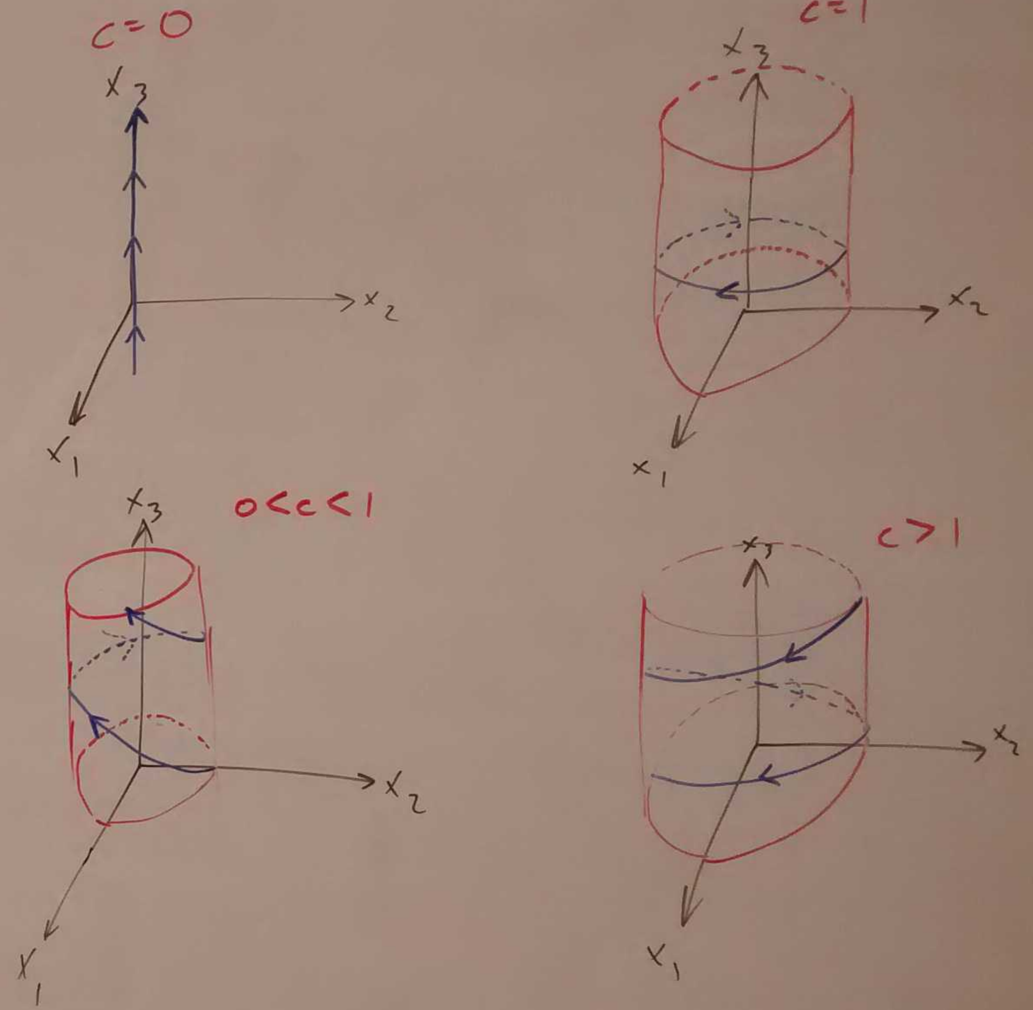
$$\dot{x}_2 = -x_1$$

$$\dot{x}_3 = 1 - (x_1^2 + x_2^2)$$

$$\frac{d}{dt}\left[x_1^2 + x_2^2\right] = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1x_2 - 2x_2x_1 = 0$$

$$\therefore \; x_1^2 + x_2^2 = k \geq 0$$

wlog $k = c^2 \quad \therefore \; x_1^2 + x_2^2 = c^2$



No fixed pts

Periodic solutions are $\begin{cases} x_1^2 + x_2^2 = 1 \\ x_3 = C \end{cases}$

Does not contradict P-B because these cylindrical regions are unbounded

$$u'' + u = f(x) \qquad u'(0) = u(1) = 0$$

$$u_1(x) = c_1 \cos x + c_2 \sin x$$
$$u_1'(x) = -c_1 \sin x + c_2 \cos x$$
$$0 = u_1'(0) = c_2 \quad , \quad c_1 := 1$$
$$\therefore u_1(x) = \cos x$$

$$u_2(x) = d_1 \cos x + d_2 \sin x$$
$$0 = u_2(1) = d_1 \cos 1 + d_2 \sin 1$$
$$d_1 := \sin 1 \Rightarrow d_2 = -\cos 1$$
$$\therefore u_2(x) = \sin 1 \cos x - \cos 1 \sin x$$
$$= \sin(1-x)$$

$$W(x) = \begin{vmatrix} \cos x & \sin(1-x) \\ -\sin x & -\cos(1-x) \end{vmatrix} = -\cos x \cos(1-x) + \sin x \sin(1-x)$$
$$= -\cos(x+1-x)$$
$$= -\cos 1$$

$$p(x) = -1 \qquad pW(x) = \cos 1$$

$$G(x,\xi) = \begin{cases} -\dfrac{u_1(x)u_2(\xi)}{p(\xi)W(\xi)} & x < \xi \\[2ex] -\dfrac{u_1(\xi)u_2(x)}{p(\xi)W(\xi)} & x > \xi \end{cases}$$

$$= \begin{cases} -\dfrac{\cos x \sin(1-\xi)}{\cos 1} & x < \xi \\[2ex] -\dfrac{\cos \xi \sin(1-x)}{\cos 1} & x > \xi \end{cases}$$

$$u(x) = \int_0^1 G(x,\xi)\, f(\xi)\, d\xi$$

F'012 #4

$Lu = u'' + xu' + 3u \qquad u(-1) = u(1) = 0$

$$\langle v, Lu\rangle = \int_{-1}^{1} (vu'' + vxu' + 3vu)\,dx$$

$$= \left[vu'\right]_{-1}^{1} + \int_{-1}^{1} (-v'u' + vxu' + 3vu)\,dx$$

$$= \left[vu' - v'u + vxu\right]_{-1}^{1} + \int_{-1}^{1} (v''u - v'xu - vu + 3vu)\,dx$$

$$= 0 + \left[vu'\right]_{-1}^{1} + \int_{-1}^{1} (v'' - v'x + 2v)u\,dx$$

$L^* = \dfrac{d^2}{dx^2} - x\dfrac{d}{dx} + 2 \qquad$ adjoint BCs: $v(-1) = v(1) = 0.$

$v(x) = 1 - x^2 \qquad L^*v = v'' - xv' + 2(1-x^2)$

$v'(x) = -2x \qquad = -2 + 2x^2 + 2 - 2x^2$

$v''(x) = -2 \qquad = 0$

$\therefore\ 1 - x^2 \in \ker L^*$

If $Lu = f$ is solvable,

$$\int_{-1}^{1} (1-x^2) f(x)\,dx = 0.$$

F'012 #5

$$\begin{cases} \dot{x} = -xy \\ \varepsilon \dot{y} = x^2 - y \end{cases}$$

<u>outer solution:</u>

$O(1)$: $\dot{x} = -xy$ and $0 = x^2 - y$

$$x^2 = y$$

$$\dot{x} = -x(x^2)$$

$$\dot{x} = -x^3$$

$$-\frac{\dot{x}}{x^3} = 1$$

$$\frac{1}{2x^2} = t + a$$

$$\frac{1}{2x_0^2} = a$$

$$2x^2 = \frac{2x_0^2}{2x_0^2 t + 1}$$

$$x^2 = \frac{x_0^2}{2x_0^2 t + 1}$$

note: as $t \to \infty$, $x \to 0$

inner solution: $t := \varepsilon \tau$

$$\begin{cases} \frac{1}{\varepsilon} x_\tau = -xy \\ y_\tau = x^2 - y \end{cases}$$

$$x_\tau = -\varepsilon x y \approx 0$$

$$y_\tau = x^2 - y$$

x approx constant

$$y_\tau + y = x^2$$

$$\frac{d}{d\tau}(e^\tau y) = e^\tau x^2$$

$$e^\tau y = e^\tau x^2 + a$$

$$y = x^2 + a e^{-\tau}$$

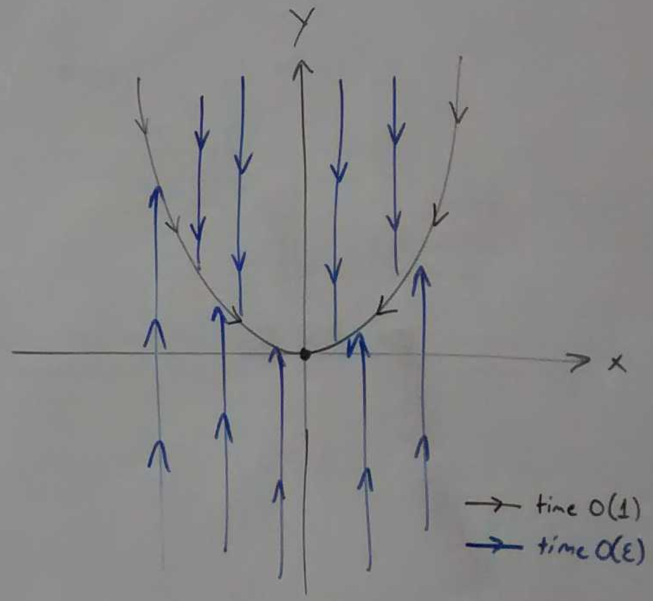
$$y_0 = x_0^2 + a$$

$$y_0 - x_0^2 = a$$

$$y(\tau) = x^2 + (y_0 - x_0^2) e^{-\tau}$$

$$\therefore y(t) = x^2 + (y_0 - x_0^2) e^{-\frac{t}{\varepsilon}}$$

Note: for $t \gg \varepsilon$, $y = x^2$



Solutions rapidly approach $y = x^2$ then approach $(0,0)$

$$u_{tt} - u_{xx} = 0$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,t) := F(x)G(t)$$

$$FG'' - F''G = 0$$

$$\frac{G''}{G} = \frac{F''}{F} := -n^2\pi^2$$

wlog $F(x) = \sin n\pi x$

$$G(t) = c_1 \cos n\pi t + c_2 \sin n\pi t$$

$$u(x,t) = \sum_{n=1}^{\infty} (c_1 \cos n\pi t + c_2 \sin n\pi t) \sin n\pi x$$

$$u_{tt} - u_{xx} = 0 \qquad \begin{aligned} u(x,0) &= f(x) \\ u(x,T) &= g(x) \\ u(0,t) &= h(t) \\ u(1,t) &= k(t) \end{aligned}$$

Let u_1, u_2 be solutions to the above, $u_1 \neq u_2$

Then $w = u_1 - u_2 \neq 0$ is a solution to

$$w_{tt} - w_{xx} = 0 \qquad \begin{aligned} w(x,0) &= 0 \\ w(x,T) &= 0 \\ w(0,t) &= 0 \\ w(1,t) &= 0 \end{aligned}$$

Have to connect this somehow to part (a)...