Problem 1. Let $f : (-1, 1) \to \mathbf{R}$ be a differentiable function such that there exists a limit

$$\lim_{x\to 0} \frac{f(x)}{x^2} = L \in \mathbf{R}$$

Does it follow that the second derivative $f''(0)$ exists and equals L? Give a proof or a counter-example.

Solution: Let f be as above. Then we must have $\lim_{x\to 0} f(x) = 0$. Therefore, by L'Hopital, $\lim_{x\to 0} \frac{f'(x)}{x} = L$. We have then that $f'(x) \to 0$ as $x \to 0$. But what is $f'(0)$?

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f(x)}{x}$$

because $\lim_{x\to 0} f(x) = 0$ and f is continuous. This limit exists since f is differentiable. We can again use L'Hopital to get that $f'(0) = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f'(x)}{1} = 0$. So $f'(0) = 0$. Therefore,

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x} = \lim_{x\to 0} \frac{f'(x)}{x} = 2L.$$

Problem 2. For functions from $[0, 1] \to \mathbf{R}$ do the following: a) Define what it means for a sequence of functions to converge uniformly. b) Explain what it means for a sequence of functions to be equicontinuous. c) Does every equicontinuous sequence of functions (that converges pointwise) converge uniformly to a continuous function? Is the converse true? Give examples or prove.

Solution: Parts (a) and (b) are just definitions. Part (c) solution: Every equicontinuous sequence of functions *that converges pointwise* must converge uniformly to a continuous function. Proof: First, if $\{f_n\}$ is equicontinuous on a compact set, then it must be uniformly equicontinuous. Let $\epsilon > 0$ be given. Proof of this fact: For all $x \in [0, 1], \exists \delta_x$ such that $|x - y| < \delta_x$ implies $|f_n(x) - f_n(y)| < \frac{\epsilon}{2}$ for all n. Cover the interval $[0, 1]$ by $\cup_{x\in[0,1]}(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$. Since $[0, 1]$ is compact, there is a finite subcover, say $(x_1 - \frac{\delta_{x1}}{2}, x_1 + \frac{\delta_{x1}}{2}), ..., (x_k - \frac{\delta_{xk}}{2}, x_k + \frac{\delta_{xk}}{2})$ that covers $[0, 1]$. Let $\delta = \frac{1}{2} \min\{\delta_{x1}, ..., \delta_{xk}\}$, then let $x, y \in [0, 1]$ such that $|x - y| < \delta$. Then $x, y \in (x_i - \frac{\delta_{xi}}{2}, x_i + \frac{\delta_{xi}}{2})$ for some i. So then:

$$|f_n(x) - f_n(y)| \le |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_n(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all n. Therefore, $\{f_n\}$ is uniformly equicontinuous.

Continuing the proof, suppose $f_n \to f$ pointwise and $\{f_n\}$ is uniformly equicontinuous. Given $\epsilon > 0$, we can pick $\delta > 0$ such that $|x-y| < \delta \Rightarrow |f_n(x)-f_n(y)| < \frac{\epsilon}{3}$ for all n. Cover $[0, 1]$ by sets of the form $(x - \delta, x + \delta)$, for all $x \in [0, 1]$. Then there exists a finite number of points x_i such that $\cup(x_i - \delta, x_i + \delta)$ covers $[0, 1]$. Since $f_n \to f$ pointwise, for each i there exists an N_i such that $m, n \ge N_i$

implies $|f_n(x_i) - f_m(x_i)| < \frac{\epsilon}{3}$. Let $N = \max_i\{N_i\}$. Let $n, m > N$, $x \in [0,1]$. Then $x \in (x_i - \delta, x_i + \delta)$ for some i. Then:

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_m(x_i)| + |f_m(x_i) - f_m(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Since $C([0,1])$ is complete with respect to the ∞-norm, f_n converges uniformly, and therefore its limit must be continuous.
Note: As printed originally, "Does every equicontinuous sequence of functions converge uniformly to a continuous function?" the answer is clearly no – just take $f_n = n$.
The converse of the modified statement is also true: If a sequence of *continuous* functions converges uniformly to a continuous function, then that sequence is equicontinuous. Proof:
Let x be given. For $\epsilon > 0$, pick N such that $n \geq N \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{3}, \forall x$. Next, let $\delta > 0$ be such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{3}$. Then for $n \geq N$, we have:

$$\begin{aligned} |f_n(x) - f_n(y)| &\leq |f_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_n(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

So for $n \geq N$, $\{f_n\}$ is equicontinuous. For $n < N$, there's only a finite number so just take the minimum δ that will work and that's it.

Problem 3. Define two sequences of functions, (f_n) and (g_n), on the interval $[0,1]$ as follows:

$$f_n(x) = (1 + cos2\pi x)^{\frac{1}{n}}, n \geq 1$$

$$g_n(x) = (1 + \frac{1}{2}cos2\pi x)^{\frac{1}{n}}, n \geq 1$$

a) What are the pointwise limits, f and g, of the sequences (f_n) and (g_n) respectively?
b) For each sequence, determine whether the convergence is uniform. Explain your answer.

Solution: a)

$$f_n(x) = \begin{cases} y^{\frac{1}{n}}, & x \neq \frac{1}{2} \\ 0, & x = \frac{1}{2} \end{cases}$$

where $y \in (0,2)$. Hence it's clear that

$$f_n \to \begin{cases} 1 & x \neq \frac{1}{2} \\ 0 & x = \frac{1}{2} \end{cases}$$

Similarly, $g_n(x) = y^{\frac{1}{n}}$ where $y \in (\frac{1}{2}, \frac{3}{2})$, so we have that $g_n \to g \equiv 1$.

b) The convergence of the f_n is not uniform. If it were, since each f_n is continuous, their limit would have to be continuous, but it is not. The convergence of the g_n is uniform. Given any $\epsilon > 0$, pick N s.t. $1 - \frac{1}{2}^{\frac{1}{N}} < \epsilon$. Then for all $x \in [0,1]$ and all $n \geq N$, $g_n(x)$ is closer to 1 than $1 - \frac{1}{2}^{\frac{1}{N}}$.

Problem 4. Let X and Y be topological spaces. Prove that if $f : X \rightarrow Y$ is continuous and X is compact, then $f(X)$ is also compact.

Solution: Let X, Y, f be as above. Let $\{Y_\alpha\}$ be an open cover of $f(X)$. Since f is continuous, $f^{-1}(Y_\alpha)$ is open for each α. The sets $f^{-1}(Y_\alpha)$ cover X, so there exists a finite subcover, say $f^{-1}(Y_1), ..., f^{-1}(Y_n)$. Then since $f(f^{-1}(Y_i)) \subset Y_i$, and $f(\cup_i f^{-1}(Y_i)) = f(X)$, we must have that $Y_1, Y_2, ..., Y_n$ form a finite subcover of $f(X)$.

Problem 5. Let X be a normed linear space and let X^* be its topological dual. Suppose that $x, y \in X$ are such that for all $\phi \in X*$, $\phi(x) = \phi(y)$. Prove that $x = y$.

Solution: Suppose $x, y \in X$ are as above, so $\phi(x) = \phi(y) \Rightarrow \phi(x-y) = 0, \forall \phi \in X^*$. The points $\{t(x-y)\}$ form a linear subspace of X. On this linear subspace we can define a functional $\lambda(t(x-y)) = t||x-y||$. By Hahn-Banach, λ can be extended to all of X. Then $\lambda \in X^*$, so $\lambda(x-y) = 0 = ||x-y|| \Rightarrow x-y = 0 \Rightarrow x = y$.

Problem 6. Consider the following equation for an unknown function $f : [0,1] \rightarrow \mathbf{R}$:

$$f(x) = g(x) + \lambda \int_0^1 (x-y)^2 f(y) dy + \frac{1}{2} \sin(f(x)) \tag{1}$$

Prove that there exists a number $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0)$, and all continuous functions g on $[0,1]$, the equation (1) has a unique continuous solution.

Solution: We will show the mapping T given by

$$Tf = g + \lambda \int_0^1 (x-y)^2 f(y) dy + \frac{1}{2} \sin(f)$$

is a contraction, and use the Contraction Mapping Theorem. So in other words, we need to show $||Tf - Th||_\infty \leq c||f-h||_\infty$, for some $c < 1$.

$$||Tf - Th||_\infty = ||\lambda \int_0^1 (x-y)^2 (f(y) - h(y)) dy + \frac{1}{2}(\sin(f(x) - h(x))||_\infty$$

The first part can be recognized as a Fredholm operator. This is easily bounded:

$$\sup_{0 \leq x \leq 1} |\int_0^1 (x-y)^2 (f(y) - h(y)) dy| \quad \leq \quad \sup_x \int_0^1 |(x-y)^2||f(y) - h(y)| dy$$

$$\begin{aligned} &\le ||f-h||_\infty \sup_{x\in[0,1]} \int_0^1 |(x-y)^2|dy \\ &\le \frac{1}{3}||f-h||_\infty \end{aligned}$$

For the other part, we have

$$\frac{\sin(f(x)) - \sin(h(x))}{f(x)-h(x)} = \cos(c) \le 1$$

for some $c \in (0,1)$ by the Mean Value Theorem, so

$$\sin(f(x)) - \sin(h(x)) \le f(x) - h(x) \qquad \forall x \in [0,1]$$

Therefore, $\frac{1}{2}(\sin(f(x)) - \sin(h(x)) \le \frac{1}{2}||f-h||_\infty$. So

$$||Tf - Th||_\infty \le \frac{\lambda}{3}||f-h||_\infty + \frac{1}{2}||f-h||_\infty$$

To have $\frac{\lambda}{3} + \frac{1}{2} < 1$, we need $\lambda < \frac{3}{2}$, so let $\lambda_0 = \frac{3}{2}$. This makes $T : C[0,1] \to C[0,1]$ a contraction, so it has a unique fixed point, i.e., so there is a unique continuous solution to (1).