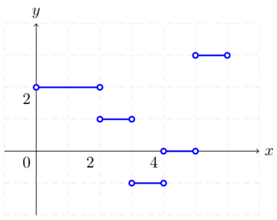
Worksheet 1 Solutions

2. Below is the graph of $f'(x)$.



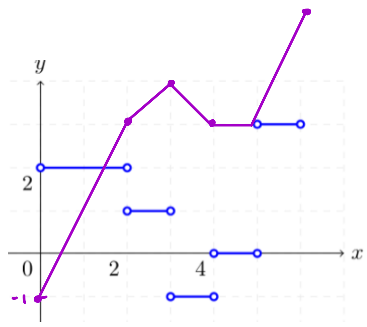
(a) Suppose that $f(x)$ is continuous on $[0,6]$ and $f(0)=-1$. Sketch the graph of $f(x)$.

(b) Suppose that $f(0)=-1$ but $f(x)$ is NOT continuous. Sketch a possible graph of $f(x)$. How many possibilities are there?

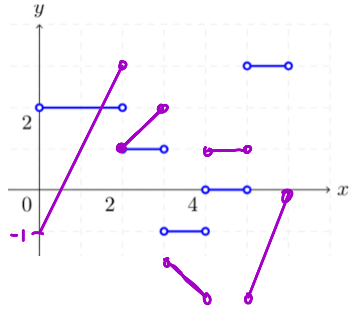
(a) Each piece of $f'(x)$ is a constant.

and the

antiderivative of $f'(x)$ is $f(x)$. So in general, $f'(x)=c$ and $f(x)=cx+b$ ← another constant. So $f(x)$ will have a slope equal to $f'(x)$ over that domain, and the initial condition and continuity condition gives us a graph:



(b) If $f(x)$ didn't have to be continuous, then we'd know the slope of each line segment but not its "+b".
There are an infinite number of possible graphs.
For example:



3. (a) Write the following definite integral as a limit:

$$\int_7^{12} 3\sin(4x)\,dx$$

(b) Write the following limit as a definite integral:

$$\lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^{n}\left[\left(3+\frac{2i}{n}\right)^2 - 3\left(3+\frac{2i}{n}\right)+4\right]$$

We know that

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$$

where $x_i = a + i\Delta x$ and $\Delta x = \frac{b-a}{n}$.

(a) Here $b=12$, $a=7$, and $f(x_i) = 3\sin(4x_i)$.

Then $\Delta x = \frac{12-7}{n} = \frac{5}{n}$, and $x_i = 7 + \frac{i5}{n}$.

Putting this all together,

$$\int_7^{12} 3\sin(4x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} 3\sin\left(4\left(7+\frac{i5}{n}\right)\right)\frac{5}{n}.$$

(b) Here we need to notice that

$$f(x_i) = \left(3+\frac{2i}{n}\right)^2 - 3\left(3+\frac{2i}{n}\right) + 4,$$

so $x_i = 3 + \frac{2i}{n}$.

We'll work backwards from $x_i = a + i\Delta x$.

Matching pieces, we find $a=3$, and $\Delta x = \frac{2}{n}$.

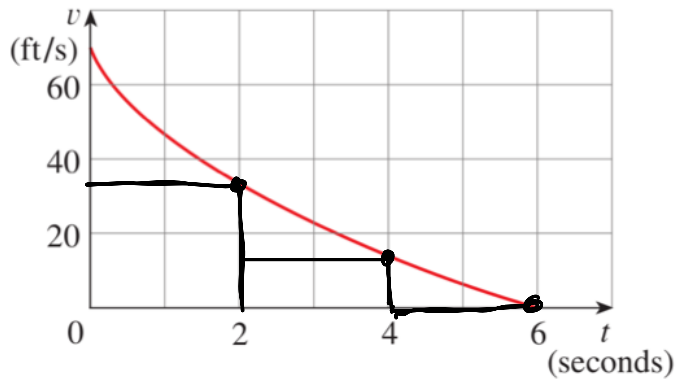
Thus $\frac{b-a}{n} = \frac{b-3}{n} = \frac{2}{n} \Rightarrow b = 5.$

So we know $\Delta x = \frac{2}{n}$, $a=3$ and $b=5$.

Thus our function must also be multiplied by 4.

$$\lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left[\left(3+\frac{2i}{n}\right)^2 - 3\left(3+\frac{2i}{n}\right)+4\right] = 4\int_3^5 x^2 - 3x + 4\,dx.$$

4. The graph below shows the *velocity* of a car as a function of time.



(a) Approximate the area under the graph for $0 \le t \le 6$ using rectangles and right endpoints.

(b) What are the *units* of your answer above? What might the area under the graph tell us physically?

(c) Is your approximation in part (a) an *over*estimate or an *under*estimate of the actual area? Would your answer change if you used a different number of rectangles?

(d) Would your answer to part (c) change if you used left endpoints instead?

(a) We're not told how many rectangles to use, thus we could technically even use 1, it would just be a worse approximation. I'll choose $n=3$. Then $\Delta t = 2$,

$$\text{Area} \approx f(2)\Delta t + f(4)\Delta t + f(6)\Delta t$$
$$\approx 35(2) + 16(2) + 0(2) = 70 + 32 = 102.$$

(b) Units are feet!
[$f(t)$ has units $\frac{ft}{s}$, and Δt has units seconds].

(c) It's extremely underestimating the area. Left endpoints
(d) would greatly overestimate the area. More rectangles would lead to a better estimate.