Problem 1) a) Fact: If $X$ is a compact space, and $f : X \to Y$ is continuous, then $f(X)$ is compact. Therefore, since $(0,1)$ is not compact, no such $f$ exists.

b) Let $f : (0,1) \to [0,1]$ be given by $f(x) = \sin(4\pi x)$. Then $f$ is both continuous and onto.

Problem 2. We let $x_i$ denote the $i$-th Fibonacci number. That is,

\begin{align*}
  x_1 &= x_2 = 1 \\
  x_{n+1} &= x_n + x_{n-1}, \quad n = 2, 3, \ldots
\end{align*}

Finally, define

$$r_n = \frac{x_{n+1}}{x_n}.$$  

Then,

$$\lim_{n \to \infty} r_n = \lim_{n \to \infty} \frac{x_{n+1}}{x_n} = \lim_{n \to \infty} \left( \frac{x_n + x_{n-1}}{x_n} \right) = \lim_{n \to \infty} \left( 1 + \frac{x_{n-1}}{x_n} \right) = 1 + \frac{1}{\lim_{n \to \infty} \frac{x_n}{x_{n-1}}}.$$  

There are two ways to finish this problem: The first uses the above with elementary analysis. The second proof uses the Contraction Mapping Theorem:

1. Continuing, we have

$$\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = 1 + \frac{1}{\lim_{n \to \infty} \frac{x_n}{x_{n-1}}}. \tag{1}$$

Let $L$ denote this quantity:\footnote{Note, we assume that this limit exists, though perhaps on an exam, we should prove this!}

$$L = \lim_{n \to \infty} \frac{x_{n+1}}{x_n}.$$  

Then one also has

$$L = \lim_{n \to \infty} \frac{x_n}{x_{n-1}}.$$  

By applying the assignments of $L$ into (1), we have

$$L = 1 + \frac{1}{L}.$$  

2. In the second proof, by rewriting the work above without limits, we have

\[ r_n = 1 + \frac{1}{r_{n-1}}. \]

Based on this formula, we define the map \( T \)

\[ T(x) = 1 + \frac{1}{x}. \quad (2) \]

By computing the first several elements of the sequence \( \{r_i\}_{i=1}^\infty \) by hand, we note that \( r_n \geq \frac{3}{2} \) for \( n \geq 3 \). Thus, we’ll take the rule (2) above for \( T \) and define \( T \) as a map on the following domain and codomain

\[ T : \left[\frac{3}{2}, \infty \right) \rightarrow \left[\frac{3}{2}, \infty \right). \]

We bound the value \( \|T(x) - T(y)\| \) for arbitrary \( x, y \in X \) as follows:

\[
\|T(x) - T(y)\| = \left\| 1 + \frac{1}{x} - \left( 1 + \frac{1}{y} \right) \right\| \\
= \left\| \frac{y - x}{xy} \right\| \\
\leq \frac{4}{9} \|x - y\|. 
\]

Since \( X = \left[\frac{3}{2}, \infty \right) \) is a complete metric space, the function \( T : X \rightarrow X \) is a contraction. By the Contraction Mapping Theorem, \( T \) has a unique fixed point \( L \in X \). That is,

\[ L = 1 + \frac{1}{L}. \]

The branching of solutions is done. In either method, one obtains \( L^2 = L + 1 \), thus \( L^2 - L - 1 = 0 \). By using the quadratic equation we get

\[ L = \frac{1 \pm \sqrt{5}}{2} \]

though the root \( \frac{1}{2}(1 - \sqrt{5}) \) is negative. We conclude (since all the terms in the Fibonacci sequence are positive) that

\[ L = \frac{1 + \sqrt{5}}{2} = \phi. \]
Problem 3) Define a sequence \((x_n)\) such that \(x_n \in F_n, \forall n\). Then \((x_n)\) is Cauchy, since given any \(\epsilon > 0\), we can pick \(N \in \mathbb{N}\) such that \(\text{diam}(F_N) < \epsilon\). Then for \(m, n \geq N\), we have \(x_n, x_m \in F_N\), so

\[
d(x_n, x_m) \leq \sup\{d(x, y) : x, y \in F_N\} \leq \epsilon
\]

Since \(X\) is complete, \((x_n)\) has a limit point \(x\) in \(X\).

Claim: \(x \in \bigcap_{n=1}^{\infty} F_n\)

Proof of claim: Suppose not. Then \(\exists M \in \mathbb{N}\) such that \(x \notin F_M\). Then \(x \notin F_n, \forall n \geq M\), by the inclusion property of the \(F_n\). The tail of \((x_n)\), for \(n \geq M\), is contained in \(F_{M+1}\). So \((x_n)\) is a sequence in \(F_{M+1}\) such that \(x_n \to x\). But \(F_{M+1}\) is closed, so \(x \in F_{M+1}\), which is a contradiction. Therefore, \(\bigcap_{n=1}^{\infty} F_n\) is nonempty.

Now to show \(x\) is unique. Suppose \(x, y \in \bigcap_{n=1}^{\infty} F_n\), and \(x \neq y\). Then \(d(x, y) = c > 0\). By definition of the \(F_n\), there is an \(N\) such that \(\text{diam}(F_N) < \epsilon\). Then not both \(x\) and \(y\) can lie in \(F_N\), which is a contradiction. Therefore, \(\bigcap_{n=1}^{\infty} F_n = \{x\}\).

Problem 4) If \(f = g\), then clearly \(f \ast f = \frac{1}{2}(f \ast f + f \ast f)\). Suppose that \(f \ast g = \frac{1}{2}(f \ast f + g \ast g)\). Then we can look at the Fourier coefficients to get \(\hat{f_n}\hat{g_n} = \frac{1}{2}(\hat{f_n}^2 + \hat{g_n}^2) \Rightarrow \hat{f_n} - \hat{g_n})^2 = 0 \Rightarrow \hat{f_n} = \hat{g_n} \Rightarrow f = g\).

Problem 5) Compact in \(\mathcal{H}\) is the same as sequentially compact because \(\mathcal{H}\) is (among other things) a metric space. So when do arbitrary subsequences of \(\{a_ku_k\}\) have convergent subsequences? Claim: need \(|a_k| \to 0\). Suppose first that \(|a_k| \nrightarrow 0\). Then \(\exists \epsilon > 0\) s.t. \(\forall N \in \mathbb{N}, \exists k > N\) s.t. \(|a_k| > \epsilon\). Then define a subsequence \(\{a_ku_k\}\) by picking \(k_i\) such that \(|a_{k_i}| > \epsilon\forall k_i\). Then:

\[
||a_{k_i}u_{k_i} - a_{k_j}u_{k_j}||^2 = \langle a_{k_i}u_{k_i} - a_{k_j}u_{k_j}, a_{k_i}u_{k_i} - a_{k_j}u_{k_j} \rangle = ||a_{k_i}u_{k_i}||^2 + ||a_{k_j}u_{k_j}||^2 - 2\langle a_{k_i}u_{k_i}, a_{k_j}u_{k_j} \rangle
\]

\[
= |a_{k_i}|^2 + |a_{k_j}|^2 > 2\epsilon^2 > 0
\]

since \(\langle u_{k_i}, u_{k_j} \rangle = 0\). So any subsequence of this sequence is not Cauchy and therefore cannot converge.

Now suppose \(|a_k| \to 0\). Let \(\{a_ku_k\}\) be an arbitrary subsequence. Then the same calculation as above shows that \(||a_{k_i}u_{k_i} - a_{k_j}u_{k_j}|| = |a_{k_i}|^2 + |a_{k_j}|^2 \to 0\). So the sequence is Cauchy and thus converges.