Problem 1) a) Fact: If X is a compact space, and $f\colon X \to Y$ is continuous, then $f(X)$ is compact. Therefore, since $(0,1)$ is not compact, no such f exists.
b) Let $f\colon (0,1) \to [0,1]$ be given by $f(x) = sin(4\pi x)$. Then f is both continuous and onto.

Problem 2. We let x_i denote the i-th Fibonacci number. That is,

$$\begin{aligned} x_1 &= x_2 = 1 \\ x_{n+1} &= x_n + x_{n-1}, \quad n = 2, 3, \ldots \end{aligned}$$

Finally, define

$$r_n = \frac{x_{n+1}}{x_n}.$$

Then,

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \left(\frac{x_n + x_{n-1}}{x_n}\right) = \lim_{n\to\infty} \left(1 + \frac{x_{n-1}}{x_n}\right) = 1 + \frac{1}{\lim_{n\to\infty} \frac{x_n}{x_{n-1}}}.$$

There are two ways to finish this problem: The first uses the above with elementary analysis. The second proof uses the Contraction Mapping Theorem:

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1. Continuing, we have

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1 + \frac{1}{\lim_{n\to\infty} \frac{x_n}{x_{n-1}}}. \tag{1}$$

   Let L denote this quantity[1]:

$$L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

   Then one also has

$$L = \lim_{n\to\infty} \frac{x_n}{x_{n-1}}.$$

   By applying the assignments of L into (1), we have

$$L = 1 + \frac{1}{L}.$$

[1]Note, we **assume** that this limit exists, though perhaps on an exam, we should prove this!!

2. In the second proof, by rewriting the work above without limits, we have
$$r_n = 1 + \frac{1}{r_{n-1}}.$$
Based on this formula, we define the map T
$$T(x) = 1 + \frac{1}{x}. \tag{2}$$
By computing the first several elements of the sequence $\{r_i\}_{i=1}^{\infty}$ by hand, we note that $r_n \geq \frac{3}{2}$ for $n \geq 3$. Thus, we'll take the rule (2) above for T and define T as a map on the following domain and codomain
$$T : \left[\frac{3}{2}, \infty\right) \to \left[\frac{3}{2}, \infty\right).$$
We bound the value $\|T(x) - T(y)\|$ for arbitrary $x, y \in X$ as follows:
$$\begin{aligned} \|T(x) - T(y)\| &= \left\| 1 + \frac{1}{x} - \left(1 + \frac{1}{y}\right) \right\| \\ &= \left\| \frac{y - x}{xy} \right\| \\ &\leq \frac{4}{9}\|x - y\|. \end{aligned}$$
Since $X = \left[\frac{3}{2}, \infty\right)$ is a complete metric space, the function $T : X \to X$ is a contraction. By the Contraction Mapping Theorem, T has a unique fixed point $L \in X$. That is,
$$L = 1 + \frac{1}{L}.$$

The branching of solutions is done. In either method, one obtains $L^2 = L + 1$, thus $L^2 - L - 1 = 0$. By using the quadratic equation we get
$$L = \frac{1 \pm \sqrt{5}}{2}$$
though the root $\frac{1}{2}(1 - \sqrt{5})$ is negative. We conclude (since all the terms in the Fibonacci sequence are positive) that
$$L = \frac{1 + \sqrt{5}}{2} = \phi.$$

Problem 3) Define a sequence (x_n) such that $x_n \in F_n, \forall n$. Then (x_n) is Cauchy, since given any $\epsilon > 0$, we can pick $N \in \mathbf{N}$ such that $diam(F_N) < \epsilon$. Then for $m, n >= N$, we have $x_n, x_m \in F_N$, so

$$d(x_n, x_m) \leq \sup\{d(x,y) : x, y \in F_N\} \leq \epsilon$$

Since X is complete, (x_n) has a limit point x in X.

Claim: $x \in \bigcap_{n=1}^{\infty} F_n$

Proof of claim: Suppose not. Then $\exists M \in \mathbf{N}$ such that $x \notin F_M$. Then $x \notin F_n$, $\forall n \geq M$, by the inclusion property of the F_n. The tail of (x_n), for $n \geq M$, is contained in F_{M+1}. So (x_n) is a sequence in F_{M+1} such that $x_n \to x$. But F_{M+1} is closed, so $x \in F_{M+1}$, which is a contradiction. Therefore, $\bigcap_{n=1}^{\infty} F_n$ is nonempty.

Now to show x is unique. Suppose $x, y \in \bigcap_{n=1}^{\infty} F_n$, and $x \neq y$. Then $d(x,y) = c > 0$. By definition of the F_n, there is an N such that $diam(F_N) < c$. Then not both x and y can lie in F_N, which is a contradiction. Therefore, $\bigcap_{n=1}^{\infty} F_n = \{x\}$.

Problem 4) If $f = g$, then clearly $f * f = \frac{1}{2}(f * f + f * f)$. Suppose that $f * g = \frac{1}{2}(f * f + g * g)$. Then we can look at the Fourier coefficients to get $\hat{f}_n \hat{g}_n = \frac{1}{2}(\hat{f}_n^2 + \hat{g}_n^2) \Rightarrow \hat{f}_n - \hat{g}_n)^2 = 0 \Rightarrow \hat{f}_n = \hat{g}_n \Rightarrow f = g$.

Problem 5) Compact in $\mathcal{H}$ is the same as sequentially compact because $\mathcal{H}$ is (among other things) a metric space. So when do arbitrary subsequences of $\{a_k u_k\}$ have convergent subsequences? Claim: need $|a_k| \to 0$. Suppose first that $|a_k| \nrightarrow 0$. Then $\exists \epsilon > 0$ s.t. $\forall N \in \mathbb{N}, \exists k > N$ s. t. $|a_k| > \epsilon$. Then define a subsequence $\{a_{k_i} u_{k_i}\}$ by picking k_i such that $|a_{k_i}| > \epsilon \forall k_i$. Then:

$$\begin{aligned} ||a_{k_i} u_{k_i} - a_{k_j} u_{k_j}||^2 &= \langle a_{k_i} u_{k_i} - a_{k_j} u_{k_j}, a_{k_i} u_{k_i} - a_{k_j} u_{k_j} \rangle \\ &= ||a_{k_i} u_{k_i}||^2 + ||a_{k_j} u_{k_j}||^2 \\ &= |a_{k_i}|^2 + |a_{k_j}|^2 > 2\epsilon^2 > 0 \end{aligned}$$

since $\langle u_{k_i}, u_{k_j} \rangle = 0$. So any subsequence of this sequence is not Cauchy and therefore cannot converge.

Now suppose $|a_k| \to 0$. Let $\{a_{k_i} u_{k_i}\}$ be an arbitrary subsequence. Then the same calculation as above shows that $||a_{k_i} u_{k_i} - a_{k_j} u_{k_j}|| = |a_{k_i}|^2 + |a_{k_j}|^2 \to 0$. So the sequence is Cauchy and thus converges.