Problem 1. A function $f : \mathbf{R^n} \to \mathbf{R^n}$ is said to be a C^∞-function if f has continuous partial derivatives of all orders.

(a) Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \exp[1/(x^2 - 1)]$ if $|x| < 1$ and $f(x) = 0$ if $|x| \geq 1$. Show that f is a C^∞-function such that $\text{supp}(f) = [-1, 1]$. (Induction and L'Hospital's rule are needed here.)

(b) For $\epsilon > 0$ and $a \in \mathbf{R}$, show that the function $g(x) = f[(x-a)/\epsilon]$ is also a C^∞-function with $\text{supp}(g) = [a - \epsilon, a + \epsilon]$.

Solution:

Problem 2. Let $f : \mathbf{R} \to \mathbf{R}$ be integrable with respect to the Lebesgue measure. Show that the function $g : [0, \infty) \to \mathbf{R}$ defined by

$$g(t) = \sup\{\int |f(x+y) - f(x)|dx : |y| < t\}$$

for $t \geq 0$ is continuous at $t = 0$.

Solution: Let $f : \mathbf{R} \to \mathbf{R}$ be integrable with respect to Lebesgue measure, and let g be given as above. Let $\epsilon > 0$ be given. Then there exists a simple function ϕ such that $\phi \leq f$ and $\int |f - \phi| < \frac{\epsilon}{2}$. Since ϕ is simple, $\phi = (k\chi_{[0,r]} +$ other indicator functions), for some $r \in \mathbf{R}$. Then

$$|f(x+y) - f(x)| \leq |f(x+y) - \phi(x+y)| + |\phi(x+y) - \phi(x)| + |\phi(x) - f(x)|$$

Pick δ such that $0 < \delta < r$. Then for $0 \leq t < \delta$, we have

$$\begin{aligned} |g(t)| &= \sup\{\int |f(x+y) - f(x)|dx : |y| \leq t\} \\ &\leq \sup\{\int |f(x+y) - \phi(x+y)| + \int |\phi(x+y) - \phi(x)| + \int |\phi(x) - f(x)| : |y| \leq \delta\} \end{aligned}$$

$\int |\phi(x+y) - \phi(x)| = 0$ for all $|y| \leq \delta$ since for y in the interval $[0, r]$, $\phi(x+y) = \phi(x)$. Also, $\int |f(x+y) - \phi(x+y)| = \int |f(x) - \phi(x)| < \frac{\epsilon}{2}$, independent of y. Therefore, the entire right hand side above is $< \epsilon$. Since $g(0) = 0$, g is continuous at 0.

Problem 3. Consider the following theorem:

Let $1 \leq p < \infty$ and $f \in L^p$, and let $\{f_n\}$ be a sequence in L^p such that $f_n \to f$ a.e. If $\lim_{n\to\infty} ||f_n||_{L^p} = ||f||_{L^p}$, then $\lim_{n\to\infty} ||f_n - f||_{L^p} = 0$.

Show by an example that this theorem is false when $p = \infty$.

Solution: Let $f_n = \chi_{[-2,-1]} + \chi_{[n,\infty)}$. Then $f_n \to f = \chi_{[-2,-1]}$ a.e. $||f_n||_\infty = 1$, $||f||_\infty = 1$, but $||f_n - f||_\infty = 1, \forall n$.

Problem 4. On $C^0([0, 1])$ consider the two norms

$$||f||_\infty = \sup_{x\in[0,1]} |f(x)|, \quad ||f||_1 = \int_0^1 |f(x)|dx.$$

Solution:

Problem 5. Let $\mathcal{H}$ be a Hilbert space. for a subset A of $\mathcal{H}$, let $A^\perp$ denote the orthogonal complement of A.

(a) Prove that for any subset A, $(A^\perp)^\perp$ is the closed linear span of A.

(b) Prove that if A is a closed convex subset of $\mathcal{H}$, then A contains a unique element of minimal norm.

Solution: (a): Let a lie in the linear span of A. By linearity of the inner product, $< a, x >= 0\ \forall x \in A^\perp$. Therefore, by the definition of $(A^\perp)^\perp$, $a \in (A^\perp)^\perp$. Now if a lies in the closed linear span of A, then by continuity of $<,>$ we also have $< a, x >= 0$ for all $x \in A$, so $a \in (A^\perp)^\perp$. So we have that the closed linear span of A is contained in $(A^\perp)^\perp$. Next, since the closed linear span of A (denoted $< A >$ from now on) is in fact closed, we have $\mathcal{H} =< A > \oplus < A >^\perp$. We have $< A >^\perp = A^\perp$ since if $< y, a >= 0$ for all $a \in A$, then $< y, a' >= 0$ for all $a' \in A$ by linearity and continuity. So $\mathcal{H} =< A > \oplus A^\perp$. Now let $a \in (A^\perp)^\perp$. Then $a = a_1 + a_2$, where $a_1 \in < A >$, and $a_2 \in A^\perp$. Since $a \in (A^\perp)^\perp$, $< a, x >= 0$ for all $x \in A^\perp$. Therefore, $< a_1, x > + < a_2, x >= 0$ for all $x \in A^\perp$. Let $x = a_2 \in A^\perp$. Then $< a_1, a_2 > + < a_2, a_2 >= 0$. Since $a_1 \in < A >$ and $a_2 \in A^\perp$, $< a_1, a_2 >= 0$. Therefore, $< a_2, a_2 >= ||a_2||^2 = 0 \Rightarrow a_2 = 0$. Therefore, $a \in < A >$. Therefore, $(A^\perp)^\perp =< A >$.

(b) Let A be closed and convex. Let $d = \inf\{||a|| : a \in A\}$. Then $\exists a_n \in A$ such that $\lim_{n\to\infty} ||a_n|| = d$, so for all $\epsilon > 0$, there is an N such that $||a_n|| \leq d + \epsilon$. Claim: a_n is Cauchy. Proof:

$$||a_n - a_m||^2 = 2||a_n||^2 + 2||a_m||^2 - ||a_n + a_m||^2$$

by the parallelogram law. Since A is convex, $\frac{a_n+a_m}{2} \in A \Rightarrow \frac{||a_n+a_m||}{2} \geq d$. So

$$\begin{aligned} ||a_n - a_m||^2 &\leq 2(d+\epsilon)^2 + 2(d+\epsilon)^2 - 4d^2 \\ &= 8d\epsilon + 4\epsilon^2 \\ &= \epsilon(8d + 4\epsilon) \end{aligned}$$

So a_n is Cauchy.

Therefore, (a_n) converges, and since A is closed, $a_n \to a \in A$. Suppose now that $||a'|| = d$. Then $||a - a'||^2 = 2||a||^2 + 2||a'||^2 - ||a + a'||^2$. $\frac{a+a'}{2} \in A \Rightarrow ||a + a'|| \geq 2d$. So then $||a - a'||^2 \leq 2d^2 + 2d^2 - 4d^2 \leq 0 \Rightarrow a = a'$.

Problem 6. Let $\mathcal{H}$ be a Hilbert space and $X = X^* \in \mathcal{B}(\mathcal{H})$ be compact and such that

$$\frac{1}{3}X^3 - X^2 + \frac{2}{3}X = 0$$

($\mathcal{B}(\mathcal{H})$ is the bounded linear operators on $\mathcal{H}$)

(a) Prove that X can be written as the sum of two orthogonal projections, i.e., there exists orthogonal projections P and Q, such that $X = P + Q$.

(b) Explain why any two orthogonal projections P and Q such that $X = P+Q$, are necessarily of finite rank?

Solution: (a)

$$\frac{1}{3}X^3 - X^2 + \frac{2}{3}X = 0 \Rightarrow X(X-1)(X-2) = 0$$

Therefore, the only nonzero eigenvalues of X are 1 and 2. The spectral theorem for compact self-adjoint operators then says that $X = P_1 + 2P_2$, where P_i is the orthogonal projection onto the i-eigenspace. This isn't exactly the right form yet, though, since $2P_2$ is not a projection. However, we can rewrite $X = (P_1 + P_2) + P_2$. Then this works, since

$$(P_1 + P_2)^2 = P_1^2 + P_1P_2 + P_2P_1 + P_2^2 = P_1 + P_2$$

using that eigenspaces have trivial intersection, so $P_iP_j = 0$ and $P_i^2 = P_i$. Therefore, $P_1 + P_2$ is a projection. Also, $((P_1 + P_2)x, y) = (x, (P_1 + P_2)y)$ since each of P_1 and P_2 is orthogonal, so $P_1 + P_2$ is an orthogonal projection. Therefore, letting $P = P_1 + P_2$, $Q = P_2$, we have $X = P + Q$.

(b) Since X only has a finite number of eigenvalues, and we know by the spectral theorem that they have finite multiplicities, and also that they form an orthonormal basis of $\mathcal{H}$, what we have is that $\mathcal{H}$ is in fact finite-dimensional. So of course any operator on $\mathcal{H}$ has finite rank.