Problem 1. A function \( f : \mathbb{R}^n \to \mathbb{R}^n \) is said to be a \( C^\infty \)-function if \( f \) has continuous partial derivatives of all orders.

(a) Consider the function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = \exp[1/(x^2 - 1)] \) if \( |x| < 1 \) and \( f(x) = 0 \) if \( |x| \geq 1 \). Show that \( f \) is a \( C^\infty \)-function such that \( \text{supp}(f) = [-1, 1] \). (Induction and L’Hospital’s rule are needed here.)

(b) For \( \varepsilon > 0 \) and \( a \in \mathbb{R} \), show that the function \( g(x) = f[(x - a)/\varepsilon] \) is also a \( C^\infty \)-function with \( \text{supp}(g) = [a - \varepsilon, a + \varepsilon] \).

Solution:

Problem 2. Let \( f : \mathbb{R} \to \mathbb{R} \) be integrable with respect to the Lebesgue measure. Show that the function \( g : [0, \infty) \to \mathbb{R} \) defined by

\[
g(t) = \sup \{ \int |f(x + y) - f(x)| \, dx : |y| < t \}
\]

for \( t \geq 0 \) is continuous at \( t = 0 \).

Solution: Let \( f : \mathbb{R} \to \mathbb{R} \) be integrable with respect to Lebesgue measure, and let \( g \) be given as above. Let \( \varepsilon > 0 \) be given. Then there exists a simple function \( \phi \) such that \( \phi \leq f \) and \( \int |f - \phi| < \frac{\varepsilon}{2} \). Since \( \phi \) is simple, \( \phi = (k\chi_{[0, r]} + \text{other indicator functions}) \), for some \( r \in \mathbb{R} \). Then

\[
|f(x + y) - f(x)| \leq |f(x + y) - \phi(x + y)| + |\phi(x + y) - \phi(x)| + |\phi(x) - f(x)|
\]

Pick \( \delta \) such that \( 0 < \delta < r \). Then for \( 0 \leq t < \delta \), we have

\[
|g(t)| = \sup \{ \int |f(x + y) - f(x)| \, dx : |y| \leq t \} \\
\leq \sup \{ \int |f(x + y) - \phi(x + y)| + \int |\phi(x + y) - \phi(x)| + \int |\phi(x) - f(x)| : |y| \leq \delta \}
\]

\[
\int |\phi(x + y) - \phi(x)| = 0 \quad \text{for all } |y| \leq \delta \quad \text{since for } y \text{ in the interval } [0, r], \quad \phi(x + y) = \phi(x).
\]

Also, \( \int |f(x + y) - \phi(x + y)| = \int |f(x) - \phi(x)| < \frac{\varepsilon}{2} \), independent of \( y \). Therefore, the entire right hand side above is \( < \varepsilon \). Since \( g(0) = 0, g \) is continuous at 0.

Problem 3. Consider the following theorem:

Let \( 1 \leq p < \infty \) and \( f \in L^p \), and let \( \{f_n\} \) be a sequence in \( L^p \) such that \( f_n \to f \) a.e. If \( \lim_{n \to \infty} ||f_n||_{L^p} = ||f||_{L^p} \), then \( \lim_{n \to \infty} ||f_n - f||_{L^p} = 0 \).

Show by an example that this theorem is false when \( p = \infty \).

Solution: Let \( f_n = \chi_{[-2, -1]} + \chi_{[n, \infty)} \). Then \( f_n \to f = \chi_{[-2, -1]} \) a.e. \( ||f_n||_{\infty} = 1, ||f||_{\infty} = 1 \), but \( ||f_n - f||_{\infty} = 1, \forall n \).

Problem 4. On \( C^0([0, 1]) \) consider the two norms

\[
||f||_{\infty} = \sup_{x \in [0, 1]} |f(x)|, \quad ||f||_1 = \int_0^1 |f(x)| \, dx.
\]
Solution:

Problem 5. Let $\mathcal{H}$ be a Hilbert space. for a subset $A$ of $\mathcal{H}$, let $A^\perp$ denote the orthogonal complement of $A$.

(a) Prove that for any subset $A$, $(A^\perp)^\perp$ is the closed linear span of $A$.

(b) Prove that if $A$ is a closed convex subset of $\mathcal{H}$, then $A$ contains a unique element of minimal norm.

Solution: (a): Let $a$ lie in the linear span of $A$. By linearity of the inner product, $<a, x> = 0 \forall x \in A^\perp$. Therefore, by the definition of $(A^\perp)^\perp$, $a \in (A^\perp)^\perp$. Now if $a$ lies in the closed linear span of $A$, then by continuity of $<,>$ we also have $<a, x> = 0$ for all $x \in A$, so $a \in (A^\perp)^\perp$. So we have that the closed linear span of $A$ is contained in $(A^\perp)^\perp$. Next, since the closed linear span of $A$ (denoted $< A >$ from now on) is in fact closed, we have $\mathcal{H} = < A > \oplus < A >^\perp$. We have $< A >^\perp = A^\perp$ since if $< y, a > = 0$ for all $a \in A$, then $< y, a' > = 0$ for all $a' \in A$ by linearity and continuity. So $\mathcal{H} = < A > \oplus A^\perp$. Now let $a \in (A^\perp)^\perp$. Then $a = a_1 + a_2$, where $a_1 \in < A >$, and $a_2 \in A^\perp$. Since $a \in (A^\perp)^\perp$, $<a, x> = 0$ for all $x \in A^\perp$. Therefore, $<a_1, x > + < a_2, x> = 0$ for all $x \in A^\perp$. Let $x = a_2 \in A^\perp$. Then $<a_1, a_2 > + <a_2, a_2 > = 0$. Since $a_1 \in < A >$ and $a_2 \in A^\perp$, $<a_1, a_2 > = 0$. Therefore, $<a_2, a_2 > = ||a_2||^2 = 0 \Rightarrow a_2 = 0$. Therefore, $a \in < A >$. Therefore, $(A^\perp)^\perp = < A >$.

(b) Let $A$ be closed and convex. Let $d = \text{inf}\{|||a|| : a \in A\}$. Then $\exists a_n \in A$ such that $\lim_{n \to \infty} ||a_n|| = d$, so for all $\epsilon > 0$, there is an $N$ such that $||a_n|| \leq d + \epsilon$. Claim: $a_n$ is Cauchy. Proof:

$$||a_n - a_m||^2 = 2||a_n||^2 + 2||a_m||^2 - ||a_n + a_m||^2$$

by the parallelogram law. Since $A$ is convex, $\frac{a_n + a_m}{2} \in A \Rightarrow \frac{||a_n + a_m||}{2} \geq d$. So

$$||a_n - a_m||^2 \leq 2(d + \epsilon)^2 + 2(d + \epsilon)^2 - 4d^2 = 8d\epsilon + 4\epsilon^2 = \epsilon(8d + 4\epsilon)$$

So $a_n$ is Cauchy.

Therefore, $(a_n)$ converges, and since $A$ is closed, $a_n \to a \in A$. Suppose now that $||a'|| = d$. Then $||a - a'||^2 = 2||a||^2 + 2||a'||^2 - ||a + a'||^2$.

$$\frac{a + a'}{2} \in A \Rightarrow ||a + a'|| \geq 2d.$$ So then $||a - a'||^2 \leq 2d^2 + 2d^2 - 4d^2 \leq 0 \Rightarrow a = a'$.

Problem 6. Let $\mathcal{H}$ be a Hilbert space and $X = X^* \in B(\mathcal{H})$ be compact and such that

$$\frac{1}{3}X^3 - X^2 + \frac{2}{3}X = 0$$

($B(\mathcal{H})$ is the bounded linear operators on $\mathcal{H}$)
(a) Prove that \( X \) can be written as the sum of two orthogonal projections, i.e., there exists orthogonal projections \( P \) and \( Q \), such that \( X = P + Q \).

(b) Explain why any two orthogonal projections \( P \) and \( Q \) such that \( X = P + Q \), are necessarily of finite rank?

Solution: (a)

\[
\frac{1}{3}X^3 - X^2 + \frac{2}{3}X = 0 \Rightarrow X(X - 1)(X - 2) = 0
\]

Therefore, the only nonzero eigenvalues of \( X \) are 1 and 2. The spectral theorem for compact self-adjoint operators then says that \( X = P_1 + 2P_2 \), where \( P_i \) is the orthogonal projection onto the \( i \)-eigenspace. This isn’t exactly the right form yet, though, since \( 2P_2 \) is not a projection. However, we can rewrite \( X = (P_1 + P_2) + P_2 \). Then this works, since

\[
(P_1 + P_2)^2 = P_1^2 + P_1P_2 + P_2P_1 + P_2^2 = P_1 + P_2
\]

using that eigenspaces have trivial intersection, so \( P_iP_j = 0 \) and \( P_i^2 = P_i \). Therefore, \( P_1 + P_2 \) is a projection. Also, \( ((P_1 + P_2)x,y) = (x,(P_1 + P_2)y) \) since each of \( P_1 \) and \( P_2 \) is orthogonal, so \( P_1 + P_2 \) is an orthogonal projection. Therefore, letting \( P = P_1 + P_2 \), \( Q = P_2 \), we have \( X = P + Q \).

(b) Since \( X \) only has a finite number of eigenvalues, and we know by the spectral theorem that they have finite multiplicities, and also that they form an orthonormal basis of \( \mathcal{H} \), what we have is that \( \mathcal{H} \) is in fact finite-dimensional. So of course any operator on \( \mathcal{H} \) has finite rank.