Problem 1. Prove or disprove: Any linear bounded operator in a complex Hilbert space can be written as a linear combination of two self-adjoint operators. (Hint: Consider first the finite-dimensional case.)

Solution: Let X be a bounded operator in a complex Hilbert space. Write: $X = \frac{1}{2}(X + X^*) - \frac{i}{2}(iX - iX^*)$. Then one can check that each of the operators $(X + X^*)$ and $(iX - iX^*)$ is self-adjoint.

Problem 2. Consider the Hilbert space $L^2[-1, 1]$.
(i) Find the orthogonal complement of the space of all polynomials. (Hint: Use the Stone-Weierstrass theorem.)
(ii) Find the orthogonal complement of the space of polynomials in x^2.

Solution: i) By Stone-Weierstrass, the polynomials are dense in $C([-1, 1])$, so the orthogonal complement of the space of polynomials is the same as the orthogonal complement of the space of continuous functions. Continuous functions are dense in L^2 with respect to the L^2 norm, so the orthogonal complement is empty.
ii) (unfinished) The orthogonal complement in $L^2([-1, 1])$ of the space of polynomials in x^2 is the same as the orthogonal complement in the space of polynomials of the space of polynomials in x^2, since polynomials are dense in L^2 by part (i). Let $P(x^2)$ = space of polynomials in x^2. Then $P(x^2)^\perp \subset \{x^2\}^\perp$. So let's find $\{x^2\}^\perp$ first. Suppose $< x^2, \sum_{i=0}^n a_i x^i >= 0$. This is the same as:

$$\sum_{i=0}^{n} a_i < x^2, x^i >= \sum_{i=0}^{n} a_i \int_{-1}^{1} x^{2+i} dx = a_0' + a_2' + \cdots + a_n' = 0$$

where $a_i' = \frac{2a_i}{2+i+1}$ and n is even, if n is odd then the sum at the end above should run from a_0' to a_{n-1}'. The above holds because terms with $i = odd$ are killed. So we have that $\{x^2\}^\perp = \{\sum_{i=0}^n a_i x^i : \sum_{i\ even} \frac{2a_i}{3+i} = 0\}$. Actually, it should be the L^2 closure of this set?
But there are things in here that are not in $P(x^2)^\perp$. For instance, $3x^6 - \frac{2}{3} \in \{x^2\}^\perp$, but $< 3x^6 - \frac{2}{3}, x^2 + 1 >\neq 0$, so $3x^6 - \frac{2}{3} \notin P(x^2)^\perp$.
...

Problem 3. Consider the space of all polynomials on $[0, 1]$ vanishing at the origin, with the sup norm. Prove that the space is not complete and find its completion.

Solution: We can approximate $\sin(x)$ by Taylor series. Every Taylor series approximation is a polynomial that is 0 at the origin. The Taylor series approximations are Cauchy, since the tails go to zero, and converge to $\sin(x)$, but $\sin(x)$ is not in the space, so the space is not complete.
Conjecture: The completion is the space of all continuous functions on $[0, 1]$ that vanish at the origin.

Problem 4. Prove that $\mathbf{R^1}$ with the metrics
(i) $\rho(x,y) = |arctan(x) - arctan(y)|$
or
(ii) $\rho(x,y) = |exp(x) - exp(y)|$
is incomplete, and find the completion in each case.

Solution: (i) Define a sequence (x_n) by $x_n = n$. Then (x_n) is Cauchy with respect to the given metric since $\arctan(n)$ gets arbitrarily close to $\frac{\pi}{2}$ as $n \to \infty$. However, (x_n) does not have a limit in $\mathbf{R}$. For, if it did and $x_n \to x \in \mathbf{R}$, we would have $|\arctan(x) - \frac{\pi}{2}| = c > 0$, and could then find N such that $n \geq N \Rightarrow |\arctan(x_n) - \frac{\pi}{2}| < \frac{c}{2}$, so that $\rho(x_n, x) > \frac{c}{2}\ \forall n \geq N$, which is a contradiction. The completion of $\mathbf{R}$ with respect to ρ is $\mathbf{R} \cup \{\pm\infty\}$. Proof?
(ii) We can do the same trick as above by setting $x_n = -n$. Then $exp(x_n) \to 0$ as $n \to \infty$, so $(x_n$ is Cauchy. But it does not converge to anything in $\mathbf{R}$ by the same reasoning as above. In this case, however, it only goes in one direction, so the completion of $\mathbf{R}$ with respect to ρ is $\mathbf{R} \cup \{-\infty\}$.

Problem 5. Consider a continuous mapping of the closed unit square $[0,1]\times[0,1]$ into some metric space X. Prove that the image of the square under such a mapping is compact.

Solution: In general, if Y is compact, and $f : Y \to X$ is continuous, then $f(Y)$ is compact. Proof sketch: Let $\{X_\alpha\}$ be a covering of $f(Y)$. Then $\{f^{-1}(X_\alpha)\}$ covers Y, so there is a finite subcover, $\{f^{-1}(X_i)\}_{i=1}^n$. Then $X_1, \ldots, X_n$ cover $f(Y)$.

Problem 6. Prove or disprove:
$C[0,1]$ with the usual sup norm is a Hilbert space. (Hint: Consider two continuous functions with disjoint supports and calculate the norm of their sum.)

Solution: If we take two continuous functions f and g with disjoint supports, then the norm of their sum is the max of their norms (we're talking sup-norm throughout). A norm is derived from an inner product if and only if it obeys the parallelogram law: $||f+g||^2 + ||f-g||^2 = 2||f||^2 + 2||g||^2$. If we take

$$f(x) = \begin{cases} 1-2x, & x \leq \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases}$$

and

$$g(x) = \begin{cases} 0 & x \leq \frac{1}{2} \\ \frac{1}{2} + \frac{x}{2} & x > \frac{1}{2} \end{cases}$$

then $||f+g||_\infty = 1$, $||f-g||_\infty = 1$, $||f||_\infty = 1$, $||g||_\infty = \frac{1}{2}$, so the parallelogram identity would say: $1^2+1^2 = 2(1^2)+2(\frac{1}{2})^2 \Rightarrow 2 = 2+\frac{1}{2}$, which is a contradiction. So it cannot be a Hilbert space.