Problem 1. Let \( f : (-1, 1) \to \mathbb{R} \) be a differentiable function such that there exists a limit
\[
\lim_{x \to 0} \frac{f(x)}{x^2} = L \in \mathbb{R}
\]
Does it follow that the second derivative \( f''(0) \) exists and equals \( L \)? Give a proof or a counter-example.

Solution: Let \( f \) be as above. Then we must have \( \lim_{x \to 0} f(x) = 0 \). Therefore, by L’Hospital, \( \lim_{x \to 0} \frac{f(x)}{x} = L \). We have then that \( f'(0) = 0 \) as \( x \to 0 \). But what is \( f''(0) \)?

\[
f''(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x)}{x}
\]
because \( \lim_{x \to 0} f(x) = 0 \) and \( f \) is continuous. This limit exists since \( f \) is differentiable. We can again use L’Hospital to get that \( f''(0) = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{f'(x)}{1} = 0 \). So \( f''(0) = 0 \). Therefore,

\[
f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x} = \lim_{x \to 0} \frac{f'(x)}{x} = 2L.
\]

Problem 2. For functions from \([0,1] \to \mathbb{R}\) do the following: a) Define what it means for a sequence of functions to converge uniformly. b) Explain what it means for a sequence of functions to be equicon- tinuous. c) Does every equicon- tinuous sequence of functions (that converges pointwise) converge uniformly to a continuous function? Is the converse true? Give examples or prove.

Solution: Parts (a) and (b) are just definitions. Part (c) solution: Every equicon- tinuous sequence of functions that converges pointwise must converge uniformly to a continuous function. Proof: First, if \( \{f_n\} \) is equicontinuous on a compact set, then it must be uniformly equicontinuous. Let \( \epsilon > 0 \) be given. Proof of this fact: For all \( x \in [0,1] \), \( \exists \delta_x \) such that \( \|x - y\| < \delta_x \) implies \( \|f_n(x) - f_n(y)\| < \frac{\epsilon}{2} \) for all \( n \). Cover the interval \([0,1]\) by \( \bigcup_{x \in [0,1]} (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \). Since \([0,1]\) is compact, there is a finite subcover, say \((x_1 - \frac{\delta_{x_1}}{2}, x_1 + \frac{\delta_{x_1}}{2}), \ldots, (x_k - \frac{\delta_{x_k}}{2}, x_k + \frac{\delta_{x_k}}{2}) \) that covers \([0,1]\). Let \( \delta = \frac{1}{2} \min \{\delta_{x_1}, \ldots, \delta_{x_k}\} \), then let \( x, y \in [0,1] \) such that \( \|x - y\| < \delta \). Then \( x, y \in (x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2}) \) for some \( i \). So then:

\[
|f_n(x) - f_n(y)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_n(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

for all \( n \). Therefore, \( \{f_n\} \) is uniformly equicontinuous.

Continuing the proof, suppose \( f_n \to f \) pointwise and \( \{f_n\} \) is uniformly equicon- tinuous. Given \( \epsilon > 0 \), we can pick \( \delta > 0 \) such that \( \|x - y\| < \delta \Rightarrow \|f_n(x) - f_n(y)\| < \frac{\epsilon}{2} \) for all \( n \). Cover \([0,1]\) by sets of the form \( (x - \delta, x + \delta) \), for all \( x \in [0,1] \). Then there exists a finite number of points \( x_i \) such that \( \cup (x_i - \delta, x_i + \delta) \) covers \([0,1]\). Since \( f_n \to f \) pointwise, for each \( i \) there exists an \( N_i \) such that \( m, n \geq N_i \)
implies $|f_n(x_i) - f_m(x_i)| < \frac{\epsilon}{3}$. Let $N = \max\{N_i\}$. Let $n, m > N$, $x \in [0, 1]$. Then $x \in (x_i - \delta, x_i + \delta)$ for some $i$. Then:

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_m(x_i)| + |f_m(x_i) - f_m(x)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

Since $C([0, 1])$ is complete with respect to the $\infty$-norm, $f_n$ converges uniformly, and therefore its limit must be continuous.

Note: As printed originally, "Does every equicontinuous sequence of functions converge uniformly to a continuous function?" the answer is clearly no – just take $f_n = n$.

The converse of the modified statement is also true: If a sequence of continuous functions converges uniformly to a continuous function, then that sequence is equicontinuous. Proof:

Let $x$ be given. For $\epsilon > 0$, pick $N$ such that $n \geq N \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{3}, \forall x$. Next, let $\delta > 0$ be such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{3}$. Then for $n \geq N$, we have:

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_n(y)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

So for $n \geq N$, $\{f_n\}$ is equicontinuous. For $n < N$, there’s only a finite number so just take the minimum $\delta$ that will work and that’s it.

Problem 3. Define two sequences of functions, $(f_n)$ and $(g_n)$, on the interval $[0, 1]$ as follows:

$$f_n(x) = (1 + \cos 2\pi x)^\frac{1}{n}, n \geq 1$$

$$g_n(x) = (1 + \frac{1}{2}\cos 2\pi x)^\frac{1}{n}, n \geq 1$$

a) What are the pointwise limits, $f$ and $g$, of the sequences $(f_n)$ and $(g_n)$ respectively?

b) For each sequence, determine whether the convergence is uniform. Explain your answer.

Solution: a)

$$f_n(x) = \begin{cases} 
  y^{\frac{1}{n}}, & x \neq \frac{1}{2} \\
  0, & x = \frac{1}{2}
\end{cases}$$

where $y \in (0, 2)$. Hence it’s clear that

$$f_n \to \begin{cases} 
  1 & x \neq \frac{1}{2} \\
  0 & x = \frac{1}{2}
\end{cases}$$

Similarly, $g_n(x) = y^{\frac{1}{n}}$ where $y \in (\frac{1}{2}, \frac{3}{2})$, so we have that $g_n \to g \equiv 1$. 

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b) The convergence of the $f_n$ is not uniform. If it were, since each $f_n$ is continuous, their limit would have to be continuous, but it is not. The convergence of the $g_n$ is uniform. Given any $\epsilon > 0$, pick $N$ s.t. $1 - \frac{1}{2}^{N} < \epsilon$. Then for all $x \in [0, 1]$ and all $n \geq N$, $g_n(x)$ is closer to 1 than $1 - \frac{1}{2}^{N}$.

Problem 4. Let $X$ and $Y$ be topological spaces. Prove that if $f : X \rightarrow Y$ is continuous and $X$ is compact, then $f(X)$ is also compact.

Solution: Let $X, Y, f$ be as above. Let $\{Y_\alpha\}$ be an open cover of $f(X)$. Since $f$ is continuous, $f^{-1}(Y_\alpha)$ is open for each $\alpha$. The sets $f^{-1}(Y_\alpha)$ cover $X$, so there exists a finite subcover, say $f^{-1}(Y_1), ..., f^{-1}(Y_n)$. Then since $f(f^{-1}(Y_i)) \subset Y_i$, and $f(\cup_i f^{-1}(Y_i)) = f(X)$, we must have that $Y_1, Y_2, ..., Y_n$ form a finite subcover of $f(X)$.

Problem 5. Let $X$ be a normed linear space and let $X^*$ be its topological dual. Suppose that $x, y \in X$ are such that for all $\phi \in X^*$, $\phi(x) = \phi(y)$. Prove that $x = y$.

Solution: Suppose $x, y \in X$ are as above, so $\phi(x) = \phi(y) \Rightarrow \phi(x - y) = 0, \forall \phi \in X^*$. The points $\{t(x - y)\}$ form a linear subspace of $X$. On this linear subspace we can define a functional $\lambda(t(x-y)) = t||x-y||$. By Hahn-Banach, $\lambda$ can be extended to all of $X$. Then $\lambda \in X^*$, so $\lambda(x-y) = 0 = ||x-y|| \Rightarrow x-y = 0 \Rightarrow x = y$.

Problem 6. Consider the following equation for an unknown function $f : [0, 1] \rightarrow \mathbb{R}$:

$$f(x) = g(x) + \lambda \int_0^1 (x - y)^2 f(y)dy + \frac{1}{2} \sin(f(x))$$  (1)

Prove that there exists a number $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0)$, and all continuous functions $g$ on $[0, 1]$, the equation (1) has a unique continuous solution.

Solution: We will show the mapping $T$ given by

$$Tf = g + \lambda \int_0^1 (x - y)^2 f(y)dy + \frac{1}{2} \sin(f)$$

is a contraction, and use the Contraction Mapping Theorem. So in other words, we need to show $||Tf - Th||_\infty \leq c||f-h||_\infty$, for some $c < 1$.

$$||Tf - Th||_\infty = ||\lambda \int_0^1 (x - y)^2 (f(y) - h(y))dy + \frac{1}{2} (\sin(f(x) - h(x)))||_\infty$$

The first part can be recognized as a Fredholm operator. This is easily bounded:

$$\sup_{0 \leq x \leq 1} \left| \int_0^1 (x - y)^2 (f(y) - h(y))dy \right| \leq \sup_x \int_0^1 |(x - y)^2||f(y) - h(y)|dy$$
\[ \leq ||f - h||_\infty \sup_{x \in [0,1]} \int_0^1 |(x - y)^2| dy \]
\[ \leq \frac{1}{3} ||f - h||_\infty \]

For the other part, we have
\[ \frac{\sin(f(x)) - \sin(h(x))}{f(x) - h(x)} = \cos(c) \leq 1 \]
for some \( c \in (0, 1) \) by the Mean Value Theorem, so
\[ \sin(f(x)) - \sin(h(x)) \leq f(x) - h(x) \quad \forall x \in [0,1] \]

Therefore, \( \frac{1}{2}(\sin(f(x)) - \sin(h(x))) \leq \frac{1}{2} ||f - h||_\infty \). So
\[ ||Tf - Th||_\infty \leq \frac{\lambda}{3} ||f - h||_\infty + \frac{1}{2} ||f - h||_\infty \]

To have \( \frac{\lambda}{3} + \frac{1}{2} < 1 \), we need \( \lambda < \frac{3}{2} \), so let \( \lambda_0 = \frac{3}{2} \). This makes \( T : C[0,1] \to C[0,1] \) a contraction, so it has a unique fixed point, i.e., so there is a unique continuous solution to (1).