Problem 1. (a) For a function $f : (a, b) \to \mathbf{R^1}$, (a, b) an open interval, state briefly but precisely:

i. What is meant by the statement: $f(x)$ is continuous at $x_0 \in (a, b)$.

ii. What is meant by the statement: $f(x)$ is continuous on (a, b).

iii. What is meant by the statement: $f(x)$ is uniformly continuous on (a, b).
(b) Prove, directly from the definition, that the function $f(x) = 1/x$ is uniformly continuous on the interval $[1, \infty)$.

Solution: Part (a) is just definitions. Part (b): Let $\epsilon > 0$ be given. Let $\delta = \epsilon$. Let $x, y \in [1, \infty)$ s.t. $|x - y| < \delta$. Then

$$|f(x) - f(y)| = |\frac{1}{x} - \frac{1}{y}| = |\frac{y - x}{xy}| \leq |y - x| < \epsilon.$$

Problem 2. Let $\{U_n\}_{n=1}^{\infty}$ be a nested sequence of open sets in a topological space X, so that $U_1 \subset U_2 \subset \cdots \subset U_n \subset U_{n+1}$. Let $x_n \in U_n \backslash U_{n-1}$. Set $U = \cup_{n=1}^{\infty} U_n$. Prove that $\{x_n\}$ does not have a subsequence that converges to a point in U.

Solution: Suppose $\{x_{n_k}\}$ converges to a point in U, say $x_{n_k} \to x$. Then $x \in U_i$ for some i, and given any neighborhood V of x, there exists N s.t. $\{x_{n_k}\} \in V$, $\forall n_k \geq N$. U_i itself is a neighborhood of x since U_i is open. Therefore, there exists N such that $n_k \geq N \Rightarrow x_{n_k} \in U_i$. But for $n_k > i$, we have $x_{n_k} \in U_{n_k} \backslash U_{n_k - 1} \subset U_{n_k} \backslash U_i$, so $x_{n_k} \notin U_i$. Therefore, there can be no convergent subsequence.

Problem 3. Let $T : (X, d) \to (X, d)$ be a contraction mapping from the metric space (X, d) to itself, so that for some $r < 1$, $d(Tx, Ty) \leq r d(x, y)\ \forall x, y \in X$. Assume that x_0 is a fixed point of this mapping. Prove that

$$d(x, x_0) \leq \frac{d(x, T(x))}{1 - r}$$

Solution: Let T, x_0 be as above. Then

$$d(T^n x, T^n x_0) = d(T^n x, x_0) \leq r^n d(x, x_0) \to 0$$

as $n \to \infty$. So $T^n x \to x_0$. Then by the triangle inequality and taking the limit,

$$\begin{aligned} d(x, x_0) &\leq d(x, Tx) + d(Tx, T^2 x) + d(T^2 x, T^3 x) + \cdots \\ &\leq d(x, Tx) + r d(x, Tx) + r^2 d(x, Tx) + \cdots \\ &= (1 + r + r^2 + \cdots) d(x, Tx) \\ &= \frac{d(x, Tx)}{1 - r} \end{aligned}$$

Problem 4. Let y, y' be two elements of a Hilbert space $\mathcal{H}$. Prove that if ¡y,x¿ = ¡y',x¿ for every $x \in \mathcal{H}$, then $y = y'$.

Solution: If $< y, x >=< y', x > \forall\, x \in \mathcal{H}$, then we have $< y - y', x >= 0 \forall\, x \in \mathcal{H}$. Let $x = y - y'$. Then $||y - y'||^2 = 0 \Rightarrow y = y'$.

Problem 5. Let L and R be the left shift operator and the right shift operator of $l^2(\mathbf{N})$ respectively. So

$$L(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots)$$

$$R(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, x_4, \cdots)$$

Find the point spectrum of L and R.

Solution: The point spectrum of L is all λ such that $L - \lambda I$ is not 1-1. Suppose $L(x_1, x_2, \cdots) = (x_2, x_3, x_4, \cdots) = \lambda(x_1, x_2, x_3, \cdots)$ for some λ. Then $\lambda x_1 = x_2, \lambda x_2 = x_3$, etc. So we have: $(x_1, x_2, x_3, \cdots) = (x_1, \lambda x_1, \lambda^2 x_1, \lambda^3 x_1, \cdots)$. In order for this sequence to lie in $l^2(\mathbf{N})$, we need:

$$\sum_{n=1}^{\infty} |x_n|^2 = \lambda^{-1} \sum_{n=1}^{\infty} |\lambda^n x_1|^2 = \lambda^{-1} x_1^2 \sum_{n=1}^{\infty} |\lambda|^{2n} < \infty$$

This is true iff $|\lambda| < 1$. So the point spectrum of L is $\{\lambda : |\lambda| < 1\}$.
Next, suppose $R(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots) = \lambda(x_1, x_2, x_3, \cdots)$. Then $0 = \lambda x_1$, and $x_1 = \lambda x_2, x_2 = \lambda x_3$, etc. If $\lambda \neq 0$, this implies $x_1 = 0 \Rightarrow x_2 = 0 \Rightarrow x_3 = 0$ and so on. Therefore, R has no nonzero eigenvalues.

Problem 6. Define the following three sequences of functions $[0, \infty) \to \mathbf{R}$:

$$(f_n)_{n=1}^{\infty} \text{ given by } f_n(x) = \begin{cases} \frac{n^{1/2}}{(x+1)^n}, & \text{if } 0 \leq x \leq n \\ 0, & \text{else} \end{cases}$$

$$(g_n)_{n=1}^{\infty} \text{ given by } g_n(x) = \begin{cases} \sin(2\pi n x), & \text{if } n \leq x \leq n+1 \\ 0, & \text{else} \end{cases}$$

$$(h_n)_{n=1}^{\infty} \text{ given by } h_n(x) = \sum_{k=1}^{\infty} \frac{k}{\sqrt{n}} \mathrm{Ind}_{[k, k+(1/n^2)]}(x).$$

Consider these sequences with each of the topologies given below and determine whether or not they converge and, if they converge, determine their limits. Explain your assertions.
a. Pointwise on $[0, +\infty)$.
b. Uniformly on $[0, +\infty)$.
c. In the norm topology of $L^2([0, +\infty))$.

d. Strongly in $L^{3/2}([0, +\infty))$.
e. Weakly in $L^{3/2}([0, +\infty))$.

Solution: (Note: the solutions in this part should be taken with a nice big grain of salt.)
a) Let $x \in (0, \infty)$ be fixed. Then eventually $n \geq x$, so $f_n(x) = \frac{n^{1/2}}{(x+1)^n} \to 0$ since $(x+1) > 1$, making the denominator blow up very fast (faster than the numerator, for sure). At $x = 0, f_n(0) = \frac{n^{1/2}}{1^n} = n^{1/2} \to \infty$. So f_n do not converge pointwise.
For g_n, if x is fixed, eventually we will have $x < n$, in which case $g_n(x) = 0$, so g_n converges to 0 pointwise.
What about h_n? Suppose first that $x \in \mathbf{N}$. Then for $n > x$, we have $h_n(x) = \frac{x}{\sqrt{n}} \to 0$. If $x \notin \mathbf{N}$, then $\exists N$ s.t. $n \geq N \Rightarrow x \notin [\lfloor x \rfloor, \lfloor x \rfloor + \frac{1}{n^2}]$ (where $\lfloor x \rfloor =$ floor of x). Therefore, $h_n(x) = 0$ for $n > N$. So $h_n \to 0$ pointwise for all x.

b) The functions f_n can't converge uniformly because they don't converge pointwise. The g_n also do not converge uniformly, for given any n, letting $x = n + \frac{1}{4}$ gives $g_n(x) = \sin(2\pi(n^2 + \frac{n}{4}) = 1$. The h_n also do not converge uniformly, since for any n, letting $x = n$ gives that $h_n(x) = \sqrt{n}$.

c) In the norm topology of L^2, we have:

$$||f_n||_{L^2} = \int_0^\infty |f_n(x)|^2 dx = \int_0^n |f_n(x)|^2 dx + \int_n^\infty |f_n(x)|^2 dx = \int_0^n \frac{n}{(x+1)^{2n}} dx \to 0.$$

$||g_n||_{L^2} = \int_n^{n+1} |\sin(2\pi n x)|^2 dx > 1$ always, so the g_n do not converge in L^2.

With the h_n, by drawing the graph of $|h_n(x)|^2$ one can see that:

$$\int_0^\infty |h_n(x)|^2 dx = \frac{1}{n}\frac{1}{n^2} + \frac{2^2}{n}\frac{1}{n^2} + \cdots + \frac{n^2}{n}\frac{1}{n^2} = \frac{n(n+1)(2n+1)}{6}\frac{1}{n^2} \to \infty$$

So h_n does not converge in L^2.

d) In $L^{3/2}$, we have:

$$||f_n||_{L^{3/2}} = \int_0^\infty |f_n(x)|^{3/2} dx = \int_0^n |f_n(x)|^{3/2} dx = \int_0^n \frac{n^{3/2}}{(x+1)^{3n/2}} dx \to 0.$$

$$||g_n||_{L^{3/2}} = \int_n^{n+1} |\sin(2\pi n x)|^{3/2} dx > 1$$

$$||h_n||_{L^{3/2}} = \frac{1^{3/2}}{n^{3/4}}\frac{1}{n^2} + \frac{2^{3/2}}{n^{3/4}}\frac{1}{n^2} + \cdots + \frac{n^{3/2}}{n^{3/4}}\frac{1}{n^2} = \frac{1 + 2^{3/2} + \cdots + n^{3/2}}{n^{11/4}} \leq \frac{n n^{3/2}}{n^{11/4}} = \frac{1}{n^{1/4}} \to 0$$

so h_n converges strongly in $L^{3/2}$.

e) Weakly in $L^{3/2}$. Let $r \in L^3$ (since $\frac{1}{3/2} + \frac{1}{3} = 1$) so $\int |r|^3 < \infty$. Then we need $\int_0^\infty f_n r \to 0$. $\int_0^\infty f_n r \leq ||f_n||_{L^{3/2}} ||r||_{L^3} \to 0$ by Hölder. (alternatively, just say strong $\Rightarrow$ weak?)
Next, $\int g_n r = \int_n^{n+1} r(x) \sin(2\pi n x) dx \leq \int_n^{n+1} r(x) \to 0$, so g_n converges weakly. Finally, since the h_n converge strongly, they also converge weakly.