Problem 1. (a) For a function \( f : (a, b) \to \mathbb{R}^1 \), \((a, b)\) an open interval, state briefly but precisely:

i. What is meant by the statement: \( f(x) \) is continuous at \( x_0 \in (a, b) \).

ii. What is meant by the statement: \( f(x) \) is continuous on \((a, b)\).

iii. What is meant by the statement: \( f(x) \) is uniformly continuous on \((a, b)\).

(b) Prove, directly from the definition, that the function \( f(x) = 1/x \) is uniformly continuous on the interval [1, \( \infty \)).

Solution: Part (a) is just definitions. Part (b): Let \( \epsilon > 0 \) be given. Let \( \delta = \epsilon \). Let \( x, y \in [1, \infty) \) s.t. \( |x - y| < \delta \). Then

\[
|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{y-x}{xy} \right| \leq |y-x| < \epsilon.
\]

Problem 2. Let \( \{U_n\}_{n=1}^{\infty} \) be a nested sequence of open sets in a topological space \( X \), so that \( U_1 \subset U_2 \subset \cdots \subset U_n \subset U_{n+1} \). Let \( x_n \in U_n \setminus U_{n-1} \). Set \( U = \cup_{n=1}^{\infty} U_n \). Prove that \{\( x_n \)\} does not have a subsequence that converges to a point in \( U \).

Solution: Suppose \{\( x_{n_k} \)\} converges to a point in \( U \), say \( x_{n_k} \to x \). Then \( x \in U_i \) for some \( i \), and given any neighborhood \( V \) of \( x \), there exists \( N \) s.t. \( \{x_{n_k}\} \in V, \forall n_k \geq N \). \( U_i \) itself is a neighborhood of \( x \) since \( U_i \) is open. Therefore, there exists \( N \) such that \( n_k \geq N \Rightarrow x_{n_k} \in U_i \). But for \( n_k > i \), we have \( x_{n_k} \in U_{n_k} \setminus U_{n_k-1} \subset U_{n_k} \setminus U_i \), so \( x_{n_k} \notin U_i \). Therefore, there can be no convergent subsequence.

Problem 3. Let \( T : (X, d) \to (X, d) \) be a contraction mapping from the metric space \((X, d)\) to itself, so that for some \( r < 1 \), \( d(Tx, Ty) \leq rd(x, y) \forall x, y \in X \). Assume that \( x_0 \) is a fixed point of this mapping. Prove that

\[
d(x, x_0) \leq \frac{d(x, T(x))}{1 - r}
\]

Solution: Let \( T, x_0 \) be as above. Then

\[
d(T^n x, T^n x_0) = d(T^n x, x_0) \leq r^n d(x, x_0) \to 0
\]
as \( n \to \infty \). So \( T^n x \to x_0 \). Then by the triangle inequality and taking the limit,

\[
d(x, x_0) \leq d(x, Tx) + d(Tx, T^2 x) + d(T^2 x, T^3 x) + \cdots \leq d(x, Tx) + rd(x, Tx) + r^2 d(x, Tx) + \cdots = (1 + r + r^2 + \cdots) d(x, Tx) = \frac{d(x, Tx)}{1 - r}
\]
Problem 4. Let \( y, y' \) be two elements of a Hilbert space \( H \). Prove that if \( \langle y, x \rangle = \langle y', x \rangle \) for every \( x \in H \), then \( y = y' \).

Solution: If \( \langle y, x \rangle = \langle y', x \rangle \) for every \( x \in H \), then we have \( \langle y - y', x \rangle = 0 \) for every \( x \in H \). Let \( x = y - y' \). Then \( ||y - y'||^2 = 0 \Rightarrow y = y' \).

Problem 5. Let \( L \) and \( R \) be the left shift operator and the right shift operator of \( l^2(N) \) respectively. So
\[
L(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots)
\]
\[
R(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)
\]
Find the point spectrum of \( L \) and \( R \).

Solution: The point spectrum of \( L \) is all \( \lambda \) such that \( L - \lambda I \) is not 1-1. Suppose \( L(x_1, x_2, \cdots) = \lambda(x_1, x_2, \cdots) = (x_2, x_3, x_4, \cdots) \) for some \( \lambda \). Then \( \lambda x_1 = x_2, \lambda x_2 = x_3, \) etc. So we have: \( (x_1, x_2, x_3, \cdots) = (x_1, \lambda x_1, \lambda^2 x_1, \lambda^3 x_1, \cdots) \). In order for this sequence to lie in \( l^2(N) \), we need:
\[
\sum_{n=1}^{\infty} |x_n|^2 = \lambda^{-1} \sum_{n=1}^{\infty} |\lambda^n x_1|^2 = \lambda^{-1} x_1^2 \sum_{n=1}^{\infty} \lambda^{2n} < \infty
\]
This is true iff \( |\lambda| < 1 \). So the point spectrum of \( L \) is \( \{ \lambda : |\lambda| < 1 \} \).

Next, suppose \( R(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots) = \lambda(x_1, x_2, x_3, \cdots) \). Then \( 0 = \lambda x_1, \) and \( x_1 = \lambda x_2, x_2 = \lambda x_3, \) etc. If \( \lambda \neq 0, \) this implies \( x_1 = 0 \Rightarrow x_2 = 0 \Rightarrow x_3 = 0 \) and so on. Therefore, \( R \) has no nonzero eigenvalues.

Problem 6. Define the following three sequences of functions \( [0, \infty) \rightarrow \mathbb{R} \):
\[
(f_n)_{n=1}^{\infty} \text{ given by } f_n(x) = \begin{cases} \frac{n^{1/2}}{(x+1)^n}, & \text{if } 0 \leq x \leq n \\ 0, & \text{else} \end{cases}
\]
\[
(g_n)_{n=1}^{\infty} \text{ given by } g_n(x) = \begin{cases} \sin(2\pi nx), & \text{if } n \leq x \leq n + 1 \\ 0, & \text{else} \end{cases}
\]
\[
(h_n)_{n=1}^{\infty} \text{ given by } h_n(x) = \sum_{k=1}^{\infty} \frac{k}{\sqrt{n}} \text{Ind}[k,k+(1/n^2)](x).
\]
Consider these sequences with each of the topologies given below and determine whether or not they converge and, if they converge, determine their limits. Explain your assertions.

a. Pointwise on \( [0, +\infty) \).
b. Uniformly on \( [0, +\infty) \).
c. In the norm topology of \( L^2([0, +\infty)) \).
d. Strongly in $L^{3/2}(0, +\infty)$.

e. Weakly in $L^{3/2}(0, +\infty)$.

Solution: (Note: the solutions in this part should be taken with a nice big grain of salt.)

a) Let $x \in (0, \infty)$ be fixed. Then eventually $n \geq x$, so $f_n(x) = \frac{n^{1/2}}{(x+1)^{3/2}} \to 0$ since $(x+1) > 1$, making the denominator blow up very fast (faster than the numerator, for sure). At $x = 0$, $f_n(0) = \frac{1}{n^{1/2}} = n^{1/2} \to \infty$. So $f_n$ do not converge pointwise.

For $g_n$, if $x$ is fixed, eventually we will have $x < n$, in which case $g_n(x) = 0$, so $g_n$ converges to 0 pointwise.

What about $h_n$? Suppose first that $x \in \mathbb{N}$. Then for $n > x$, we have $h_n(x) = \frac{x}{n^{1/2}} \to 0$. If $x \not\in \mathbb{N}$, then $\exists N$ s.t. $n \geq N \Rightarrow x \not\in \{[x], [x] + \frac{1}{n}\}$ (where $[x]$ = floor of $x$). Therefore, $h_n(x) = 0$ for $n > N$. So $h_n \to 0$ pointwise for all $x$.

b) The functions $f_n$ can’t converge uniformly because they don’t converge pointwise. The $g_n$ also do not converge uniformly, for given any $x$, letting $x = n + \frac{1}{4}$ gives $g_n(x) = \sin(2\pi(n^2 + \frac{1}{4})) = 1$. The $h_n$ also do not converge uniformly, since for any $n$, letting $x = n$ gives that $h_n(x) = \sqrt{n}$.

c) In the norm topology of $L^2$, we have:

$$\|f_n\|_{L^2} = \int_0^\infty |f_n(x)|^2dx = \int_0^n |f_n(x)|^2dx + \int_n^\infty |f_n(x)|^2dx = \int_0^n \frac{n}{(x+1)^{3n/2}}dx \to 0.$$  

$$\|g_n\|_{L^2} = \int_0^\infty |f_n(x)|^2dx > 1$$ always, so the $g_n$ do not converge in $L^2$.

With the $h_n$, by drawing the graph of $|h_n(x)|^2$ one can see that:

$$\int_0^\infty |h_n(x)|^2dx = \frac{1}{n} \frac{1}{n^2} + \frac{2^2}{n} \frac{1}{n^2} + \cdots + \frac{n^2}{n} \frac{1}{n^2} = \frac{n(n+1)(2n+1)}{6} \frac{1}{n^2} \to \infty$$

So $h_n$ does not converge in $L^2$.

d) In $L^{3/2}$, we have:

$$\|f_n\|_{L^{3/2}} = \int_0^\infty |f_n(x)|^{3/2}dx = \int_0^n |f_n(x)|^{3/2}dx + \int_n^\infty |f_n(x)|^{3/2}dx = \int_0^n \frac{n^{3/2}}{(x+1)^{3n/2}}dx \to 0.$$  

$$\|g_n\|_{L^{3/2}} = \int_0^\infty |f_n(x)|^{3/2}dx > 1$$

$$\|h_n\|_{L^{3/2}} = \int_0^\infty |f_n(x)|^{3/2}dx = \frac{1}{n^{3/4}} \frac{1}{n^2} + \frac{2^3/2}{n} \frac{1}{n^2} + \cdots + \frac{n^{3/2}}{n} \frac{1}{n^2} = \frac{1}{n^{11/4}} \sum_{n=1}^\infty \frac{1}{n^{1/4}} \leq \frac{nn^{3/2}}{n^{11/4}} = \frac{1}{n^{1/4}} \to 0$$

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so $h_n$ converges strongly in $L^{3/2}$.

e) Weakly in $L^{3/2}$. Let $r \in L^3$ (since $\frac{1}{3} + \frac{1}{3} = 1$) so $\int |r|^3 < \infty$. Then we need $\int_0^\infty f_n r \to 0$. $\int_0^\infty f_n r \leq \|f_n\|_{L^{3/2}} \|r\|_{L^3} \to 0$ by Hölder. (alternatively, just say strong $\Rightarrow$ weak?)

Next, $\int g_n r = \int_{n+1}^{n+1} r(x) \sin(2\pi nx)dx \leq \int_{n}^{n+1} r(x) \to 0$, so $g_n$ converges weakly. Finally, since the $h_n$ converge strongly, they also converge weakly.